Appl. Math. Mech. -Engl. Ed., 2008, **29**(7):889–896
DOI 10.1007/s10483-008-0707-x
©Shanghai University and Springer-Verlag 2008

Applied Mathematics and Mechanics (English Edition)

Large deflection of circular membrane under concentrated force *

JIN Cong-rui

(Department of Mechanical Engineering, University of Alberta, AB, T6G 2G8, Canada)

(Communicated by CHENG Chang-jun)

Abstract The analytical solution to a Föppl-Hencky membrane with a rigidly clamped boundary condition under concentrated force is provided. Stability of a nonlinear circular membrane is investigated.

Key words circular membrane, concentrated force, large deformation, exact solution

Chinese Library Classification O344.03
2000 Mathematics Subject Classification 74B20, 74K35

Introduction

The problem of the axisymmetric large deformation of a circular membrane with a rigidly clamped boundary condition under a concentrated force is of great interest. Chen & Zheng[1] gave a solution using the Hencky transformation. In this paper, we show that some of their solutions are not wrinkle-free solutions, i.e., the membrane goes into a special asymmetric mode of deformation.

1 Governing equations

The governing equations for a Föppl-Hencky membrane under the concentrated force are

$$r\frac{d}{dr}\Big[\frac{d}{rdr}(r^2N_r)\Big] = -\frac{hE}{2}\Big(\frac{dw}{dr}\Big)^2, \tag{1}$$

$$N_r\frac{dw}{dr} = -\frac{P_0}{2\pi r}, \tag{2}$$

with the rigidly clamped boundary condition $w = 0$ and

$$u = \frac{a}{Eh}[a\frac{dN_r}{dr} + (1-\nu)N_r] = 0 \tag{3}$$

at $r = a$, where r is the radial position, a is the radius of the circular membrane, h is the thickness, w is the deflection, N_r is the radial membrane force, E is Young's modulus, ν is Poisson's ratio and P_0 is the concentrated force.

We introduce the following variables:

$$y = \frac{r^2}{a^2}, \quad \phi = \frac{r}{h}\frac{dw}{dr}, \quad S = \frac{8r^2N_r}{Eh^3}, \quad F = \frac{4a^2P_0}{\pi Eh^4}. \tag{4}$$

* Received Dec. 1, 2007 / Revised Jun. 8, 2008
Corresponding author JIN Cong-rui, E-mail: congrui@ualberta.ca

Then we get the dimensionless forms of Eqs. (1) and (2),

$$\frac{d^2S}{dy^2} = -\frac{\phi^2}{y^2}, \tag{5}$$

$$S\phi + Fy = 0. \tag{6}$$

Deleting ϕ, we obtain

$$\frac{d^2S}{dy^2} + \frac{F^2}{S^2} = 0. \tag{7}$$

The rigidly clamped boundary condition becomes

$$2\frac{dS}{dy} - (1+\nu)S = 0 \ \text{ at } \ y = 1. \tag{8}$$

At the center of the circular membrane, due to the concentrated force which induces N_r with singularity, we have

$$\lim_{y\to 0} S = O(y^\alpha), \ \ \alpha < 1. \tag{9}$$

Equation (7) is a nonlinear second-order differential equation whose integration yields two unknown constants determined not only by the boundary conditions but also by the stability conditions. From physical reasons, it follows that the tensile solutions may open the possibility of wrinkling, i.e., a special asymmetric mode of deformation to which the Föppl-Hencky equations do not apply any more, if N_r or N_θ is negative in parts of the membrane.

2 Analytical solutions

From Eq. (7), we immediately have the first integral

$$\frac{1}{2}\Big(\frac{dS}{dy}\Big)^2 = \frac{F^2}{S} - C_1, \tag{10}$$

where C_1 is an undetermined integration constant.

Equation (10) gives

$$\frac{dS}{dy} = \pm\Big[\frac{2(F^2 - C_1S)}{S}\Big]^{1/2}. \tag{11}$$

This shows that the solution must be considered in three cases defined by the integration constant C_1, namely, positive, zero and negative.

Case 1 $C_1 = 0$

Integrating both sides of Eq. (11) gives

$$S = \Big(\frac{9}{2}F^2\Big)^{1/3}(\pm y + C_2)^{2/3}, \tag{12}$$

where C_2 is another integration constant.

Substituting Eq. (12) into Eq. (6) gives

$$\phi = -\Big(\frac{2}{9}F\Big)^{1/3}(\pm y + C_2)^{-2/3}y. \tag{13}$$

We get the deformed film profile,

$$\frac{w}{h} = \Big(\frac{3}{4}F\Big)^{1/3}[(\pm 1 + C_2)^{1/3} - (\pm y + C_2)^{1/3}]. \tag{14}$$

Physical meaning requires w to be positive. Thereby, we should adopt the positive sign in Eq. (14), which means that Eqs. (11)–(13) should take the positive signs. The central deflection is, therefore,

$$\frac{w_0}{h} = \left(\frac{3}{4}F\right)^{\frac{1}{3}}[(1+C_2)^{\frac{1}{3}} - C_2^{\frac{1}{3}}]. \tag{15}$$

The radial stress is obtained from Eq. (12),

$$N_r = \frac{9^{1/3}}{4}\left(\frac{P_0^2 Eh}{a^2\pi^2}\right)^{1/3}\frac{(y+C_2)^{2/3}}{y}, \tag{16}$$

which is always nonnegative. At the membrane center $y = 0$, N_r becomes singular, meaning that the result is in accord with the condition that the concentrated force induces N_r with singularity.

Since we have

$$N_\theta = N_r + r\frac{dN_r}{dr}, \tag{17}$$

substituting Eq. (16) into Eq. (17) gives

$$N_\theta = \frac{9^{1/3}}{4}\left(\frac{P_0^2 Eh}{a^2\pi^2}\right)^{1/3}(y+C_2)^{-1/3}\left(\frac{1}{3} - \frac{C_2}{y}\right), \tag{18}$$

which shows that the nonnegative hoop stress requires that $C_2 = 0$.

Substituting Eq. (12) into the rigidly clamped boundary Eq. (8) and making use of $C_2 = 0$, we have

$$\frac{1-3\nu}{3(1+\nu)} = 0, \tag{19}$$

which gives $\nu = \frac{1}{3}$. Substituting $C_2 = 0$ into Eq. (15) gives

$$\frac{w_0}{h} = \left(\frac{3}{4}F\right)^{1/3}. \tag{20}$$

Case 2 $C_1 > 0$

We introduce a new variable θ such that

$$S = \frac{F^2}{C_1}\sin^2\theta, \tag{21}$$

where $0 \le \theta \le \frac{\pi}{2}$. Substituting Eq. (21) into Eq. (11), we obtain

$$\frac{d\theta}{dy} = 2^{-\frac{1}{2}}C_1^{\frac{3}{2}}F^{-2}\csc^2\theta. \tag{22}$$

After integrating, we obtain

$$y + C_2 = (2C_1)^{-\frac{3}{2}}F^2(2\theta - \sin 2\theta), \tag{23}$$

where C_2 is another undetermined integration constant.

Substituting Eq. (21) into Eq. (6) gives

$$\phi = -\frac{C_1 y}{F\sin^2\theta}. \tag{24}$$

By resorting to Eq. (4), we get the deformed film profile

$$\frac{w}{h} = -\int_1^y \frac{C_1}{2F\sin^2\theta}dy. \tag{25}$$

Substituting Eq. (22) into Eq. (25) gives

$$\frac{w}{h}=\int_{\theta(y)}^{\theta(y=1)}(2C_1)^{-\frac{1}{2}}Fd\theta=(2C_1)^{-\frac{1}{2}}F[\theta(y=1)-\theta(y)]. \tag{26}$$

The central deflection is, therefore,

$$\frac{w_0}{h}=(2C_1)^{-\frac{1}{2}}F[\theta(y=1)-\theta(y=0)]. \tag{27}$$

The exact central deflection can be calculated once C_1 and C_2 are determined from the boundary condition and the stability condition.

The radial stress is obtained from Eqs. (4) and (21),

$$N_r=\frac{Eh^3F^2}{8C_1a^2}\frac{\sin^2\theta}{y}, \tag{28}$$

which is always nonnegative and is in accord with the condition that the concentrated force induces N_r with singularity at $y=0$.

Substituting Eq. (28) into Eq. (17) gives

$$N_\theta=\frac{Eh^3F^2}{8C_1a^2}\Big[\frac{(2C_1)^{\frac{3}{2}}}{F^2}\frac{\cos\theta}{\sin\theta}-\frac{\sin^2\theta}{y}\Big]. \tag{29}$$

To ensure that N_θ in the membrane is nonnegative, we have

$$\frac{(2C_1)^{\frac{3}{2}}}{F^2}\geq\frac{\sin^3\theta}{\cos(\theta y)}. \tag{30}$$

If $y=0$, Eq. (30) gives

$$F^2\sin^3\theta\leq 0. \tag{31}$$

Because F is always positive and $0\leq\theta\leq\frac{\pi}{2}$, we have $\theta=0$ at $y=0$. Substituting this result into Eq. (23), we obtain $C_2=0$. Therefore, $C_2=0$ is the only possibility for wrinkle-free solutions.

For the rigidly clamped boundary condition, denoting $\theta(y=1)=\theta_m$ and substituting Eq. (21) into Eq. (8) give

$$\frac{(2C_1)^{\frac{3}{2}}}{F^2}=(1+\nu)\frac{\sin^3\theta_m}{\cos\theta_m}, \tag{32}$$

which gives

$$(2C_1)^{-\frac{1}{2}}F=F^{\frac{1}{3}}(1+\nu)^{-\frac{1}{3}}\frac{\cos^{\frac{1}{3}}\theta_m}{\sin\theta_m}. \tag{33}$$

Equation (27) gives

$$\frac{w_0}{h}=(2C_1)^{-\frac{1}{2}}F\theta_m. \tag{34}$$

Denote

$$\Big(\frac{w_0}{h}\Big)^3=gF. \tag{35}$$

Substituting Eq. (33) into Eq. (34) to delete C_1, we have

$$g=\frac{1}{(1+\nu)}\frac{\cos\theta_m}{\sin^3\theta_m}(\theta_m-\theta_n)^3. \tag{36}$$

Substituting Eq. (32) into Eq. (23) gives

$$(1+\nu)(y+C_2)=\frac{\cos\theta_m}{\sin^3\theta_m}(2\theta-\sin 2\theta). \tag{37}$$

At $y=1$, we have

$$(1+\nu)(1+C_2)=\frac{\cos\theta_m}{\sin^3\theta_m}(2\theta_m-\sin 2\theta_m), \tag{38}$$

which gives

$$\nu=\frac{(2\theta_m-\sin 2\theta_m)\cos\theta_m}{(1+C_2)\sin^3\theta_m}-1. \tag{39}$$

Case 3 $C_1<0$

By denoting $C_1=-\overline{C}_1$ where $\overline{C}_1$ is positive, Eq. (11) becomes

$$\frac{dS}{dy}=\Big[\frac{2(F^2+\overline{C}_1 S)}{S}\Big]^{\frac{1}{2}}. \tag{40}$$

We introduce a new variable θ such that

$$S=\frac{F^2}{\overline{C}_1}\cot^2\theta, \tag{41}$$

where $0\le\theta\le\frac{\pi}{2}$.

Substituting Eq. (41) into Eq. (40), we obtain

$$\frac{d\theta}{dy}=-2^{-\frac{1}{2}}\overline{C}_1^{\frac{3}{2}}F^{-2}\sin^3\theta\sec^2\theta. \tag{42}$$

After integrating, we obtain

$$y+\overline{C}_2=2^{-\frac{1}{2}}\overline{C}_1^{-\frac{3}{2}}F^2\Big[\frac{\cos\theta}{\sin^2\theta}-\ln\Big(\cot\frac{\theta}{2}\Big)\Big], \tag{43}$$

where $\overline{C}_2$ is another undetermined integration constant.

Substituting Eq. (41) into Eq. (6) gives

$$\phi=-\frac{\overline{C}_1 y}{F\cot^2\theta}. \tag{44}$$

By resorting to Eq. (4), we get the deformed film profile

$$\frac{w}{h}=-\int_1^y\frac{\overline{C}_1}{2F\cot^2\theta}dy. \tag{45}$$

Substituting Eq. (42) into Eq. (45) gives

$$\frac{w}{h}=\int_{\theta(y=1)}^{\theta(y)}(2\overline{C}_1)^{-\frac{1}{2}}\frac{F}{\sin\theta}d\theta=(2\overline{C}_1)^{-\frac{1}{2}}F\ln|\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}||_{\theta(y=1)}^{\theta(y)}. \tag{46}$$

The central deflection is, therefore,

$$\frac{w_0}{h}=(2\overline{C}_1)^{-\frac{1}{2}}F\ln|\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}||_{\theta(y=1)}^{\theta(y=0)}. \tag{47}$$

The radial stress is obtained from Eq. (4) and Eq. (41)

$$N_r = \frac{Eh^3F^2}{8\overline{C}_1a^2}\frac{\cot^2\theta}{y}, \tag{48}$$

which is always nonnegative and is in accord with the condition that the concentrated force induces N_r with singularity at $y = 0$.

Substituting Eq. (48) into Eq. (17) gives

$$N_\theta = \frac{Eh^3F^2}{8\overline{C}_1a^2}\Big[\frac{(2\overline{C}_1)^{\frac{3}{2}}}{F^2}\frac{1}{\cos\theta} - \frac{\cos^2\theta}{y\sin^2\theta}\Big]. \tag{49}$$

To ensure that N_θ is nonnegative, we have

$$\frac{(2\overline{C}_1)^{\frac{3}{2}}}{F^2} \geq \frac{\cos^3\theta}{\sin^2(\theta y)}. \tag{50}$$

If $y = 0$, Eq. (50) gives

$$F^2\cos^3\theta \leq 0. \tag{51}$$

Because F is always positive and $0 < \theta < \frac{\pi}{2}$, we have $\theta = \frac{\pi}{2}$ at $y = 0$. Substituting this result into Eq. (43), we obtain

$$\overline{C}_2 = 0. \tag{52}$$

For the rigidly clamped boundary condition, substituting Eq. (41) into Eq. (8), we have

$$\frac{(2\overline{C}_1)^{\frac{3}{2}}}{F^2} = (1+\nu)\frac{\cos^3\theta_m}{\sin^2\theta_m}. \tag{53}$$

Equation (47) gives

$$\frac{w_0}{h} = -(2\overline{C}_1)^{-\frac{1}{2}}F\ln\Big|\frac{1}{\sin\theta_m} - \frac{\cos\theta_m}{\sin\theta_m}\Big|. \tag{54}$$

Substituting Eq. (53) into Eq. (54) gives

$$g = -\frac{1}{(1+\nu)}\frac{\sin^2\theta_m}{\cos^3\theta_m}\ln^3\Big|\frac{1}{\sin\theta_m} - \frac{\cos\theta_m}{\sin\theta_m}\Big|. \tag{55}$$

Substituting Eq. (53) into Eq. (43) to delete C_1 gives

$$\nu = \frac{2\Big[\cos\theta_m - \sin^2\theta_m\ln\Big(\cot\frac{\theta_m}{2}\Big)\Big]}{\cos^3\theta_m} - 1. \tag{56}$$

There are some relationships among the three cases. Since $\sin\theta = \theta + \frac{\theta^3}{3!} + o(\theta^3)$, by assuming $2\theta - \sin 2\theta = \frac{(2\theta)^3}{3!}$ for small θ, we obtain from Eq. (23)

$$\theta^3 = \frac{3}{4}(y + C_2)(2C_1)^{\frac{3}{2}}F^{-2}. \tag{57}$$

By neglecting $o(\theta^3)$, Eq. (21) becomes

$$S = \frac{F^2}{C_1}\theta^2. \tag{58}$$

Getting θ^2 from Eq. (57), and then substituting it into Eq. (58), we have

$$S = \left(\frac{9}{2}F^2\right)^{\frac{1}{3}}(y + C_2)^{\frac{2}{3}}, \tag{59}$$

which is exactly the same as Eq. (12). This means that Case 2 ($C_1 > 0$) reduces to Case 1 ($C_1 = 0$) in the limit that θ is significantly small. Similarly, we can prove that Case 3 ($C_1 < 0$) reduces to Case 1 ($C_1 = 0$) in the limit that $\theta \to \frac{\pi}{2}$.

3 Conclusion

In the above analysis, the exact solutions for the nonlinear large deflection of the thin circular membrane loaded by a central point force are provided. The results show that the central deflection is proportional to the cubic root of the load with a proportional coefficient determined by Poisson's ratio ν,

$$\left(\frac{w_0}{h}\right)^3 = g(\nu)F. \tag{60}$$

For $\nu = 1/3$, $g(\nu) = 3/4$ is obtained in the $C_1 = 0$ branch. For $0 < \nu < 1/3$, $g(\nu)$ is obtained in the $C_1 > 0$ branch by combining the following two equations:

$$\nu = \frac{\cos\theta_m}{\sin^3\theta_m}(2\theta_m - \sin(2\theta_m)) - 1, \tag{61}$$

$$g = \frac{1}{(1+\nu)}\frac{\theta_m^3\cos\theta_m}{\sin^3\theta_m}. \tag{62}$$

An easy way to find $g(\nu)$ is to generate a parametric plot of $\nu(\theta_m)$ from (61) vs. $g(\nu,\theta_m)$ from (62) by varying the parameter θ_m.

For $1/3 < \nu < 1/2$, $g(\nu)$ is obtained in the $C_1 < 0$ branch by the following equations:

$$\nu = \frac{2\Big[\cos\theta_m - \sin^2\theta_m \ln\Big(\cot\frac{\theta_m}{2}\Big)\Big]}{\cos^3\theta_m} - 1, \tag{63}$$

$$g = -\frac{1}{(1+\nu)}\frac{\sin^2\theta_m}{\cos^3\theta_m}\ln^3\left|\frac{1}{\sin\theta_m} - \frac{\cos\theta_m}{\sin\theta_m}\right|. \tag{64}$$

Similarly, we generate a parametric plot of $\nu(\theta_m)$ from (63) vs. $g(\nu,\theta_m)$ from (64) by varying the parameter θ_m to find $g(\nu)$.

Komaragiri et al.[5] obtained the following approximate membrane solution for the point loads by using numerical experiments with a shooting method:

$$\left(\frac{w_0}{h}\right)^3 = \frac{\pi}{4}f^3(\nu)F, \tag{65}$$

where $f(\nu) \approx 1.049\,1 - 0.146\,2\nu - 0.158\,27\nu^2$. In Ref. [1], the stability condition is not discussed. And consequently, not all the solutions are wrinkle-free, which are given as

$$\left(\frac{w_0}{h}\right)^3 = \Big[1 - \Big(\frac{1-3\nu}{4}\Big)^{1/3}\Big]^3 \frac{F}{1+\nu}. \tag{66}$$

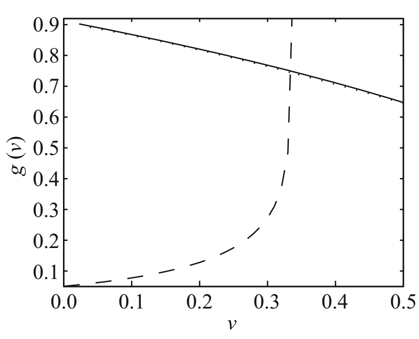

Fig. 1 $g(\nu)$ for the rigidly clamped boundary condition for $0 < \nu < 0.5$ (Solid line: exact solution described in Eqs. (61)–(64); dotted line: result from numerical simulation Eq. (65); dashed line: result from Eq. (66))

Figure 1 plots the three solutions for comparison, which shows that the result obtained by numerical experiments agrees very well with the wrinkle-free solutions, and the solution Eq. (66) is valid only for $\nu = \frac{1}{3}$.

References

[1] Chen S, Zheng Z. Large deformation of circular membrane under the concentrated force[J]. *Appl Math Mech -Engl Ed*, 2003, **24**(1):28–31. DOI:10.1007/BF02439374

[2] Weinitschke H J. Stable and unstable axisymmetric solutions for membranes of revolution[J]. *Appl Mech Rev*, 1989, **42**(11):289–294.

[3] Steigmann D J. Proof of a conjecture in elastic membrane theory[J]. *ASME J Appl Mech*, 1986, **53**(4):955–956.

[4] Beck A, Grabmüller H. Wrinkle-free solutions of circular membrane problems[J]. *Z Angew Math Phys*, 1992, **43**(3):481–504.

[5] Komaragiri U, Begley M R, Simmonds J G. The mechanical response of freestanding circular elastic films under point and pressure loads[J]. *ASME J Appl Mec*, 2005, **72**(2):203–212.