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LOCALIZED LOADS APPLIED TO TORISPHERICAL SHELLS *

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Torispherical shells are frequently employed as end closures for cylindrical shells. Such structures consist of a spherical cap joined to a toroidal segment, which is joined in turn to a cylindrical shell. In some cases, nozzles are attached to the torispherical shell. Solutions for spherical and cylindrical shells are known and approximate solutions for the toroidal shells are known.

Analyses were derived to determine stresses in a torispherical shell which is attached to a cylindrical shell and which has a radial nozzle attached at its apex. The loadings considered were internal pressure, axial thrust applied to the nozzle, and a bending moment applied to the nozzle. A particular solution was developed for the bending analysis which had no limitations such as those placed on the earlier approximate solutions.

Comparisons between the developed analyses and experimental data from two models tested under internal pressure show excellent agreement. Since the analyses for external loads are very similar to that for internal pressure, it can be assumed that the other analyses are correct also.

The buckling of torispherical shells under internal pressure was studied by comparisons between calculated results and results from an experimental model that buckled. This study indicated that the structure failed by plastic collapse since the maximum stress was very close to the yield stress of the model material.

1. Introduction

Torispherical shells are frequently employed as end closures for cylindrical shells, both in missile design and in a wide variety of industrial-type pressure vessels. Such vessels consist of a spherical cap joined to a toroidal segment, joined in turn to a cylindrical shell. In some cases, nozzles are attached to the torispherical shell.

Since the solutions for spherical and cylindrical shells are known, it is only necessary to derive a solution for the toroidal segment. There have been several approximate solutions derived for the bending of a toroidal shell segment under axisymmetric loading–notably, the work done by Clark, Reissner, and Novozhilov. There have been several numerical methods used to get solutions for the torus, but, in general, these methods have the same approximations as the analytical solutions and therefore no better accuracy is gained [1].

One of the difficulties encountered in the toroidal shell is that the membrane solution is invalid at $\phi = 0^\circ$ or 180° (see fig. 1). Fortunately, in most pressure vessels and in the torispherical shell, the range of ϕ is between 90° and ϕ_0, the angle at the top of the toroidal shell, where ϕ_0 is greater than 0°. This allows the use of an analytical solution for the analysis of these pressure vessel configurations. Another difficulty encountered is that the solution from the membrane theory is not the particular solution to the bending equations. The usual particular solution for the bending equation is based on the membrane theory but some exceptions to this are the toroidal and ellipsoidal shells. The interaction between the bending and membrane effects increases the difficulty of the solution and this is the area where most of the limiting approximations or assumptions are made. One of the purposes of this investigation is to develop a new method for finding the particular solution in the case of axisymmetric loading and de-

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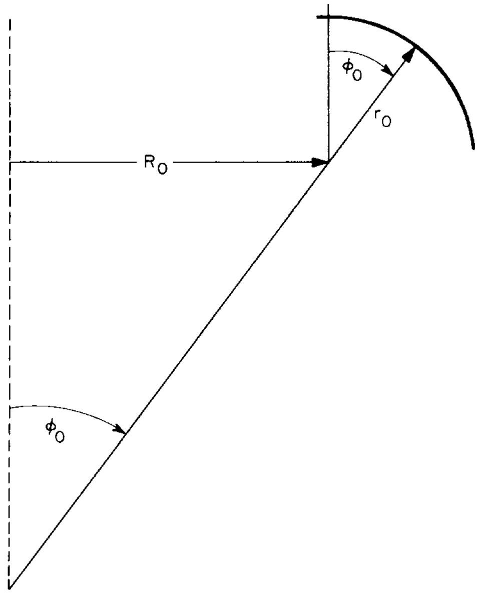

Fig. 1. Geometry of the toroidal shell.

velop a method to find the particular solution in the case of nonsymmetrical loading. Particular attention was directed to the case of a torispherical shell loaded by a concentrated moment or thrust at the apex. With this solution, the torispherical shell with a radial nozzle attached at the apex can be analyzed for both an external moment applied to the nozzle and an axial thrust on the nozzle. This configuration is of much interest to industry.

An experimental study has shown that torispherical shells can buckle from internal pressure due to high circumferential stresses [2]. Such studies were undertaken after vessel heads collapsed while undergoing their hydrostatic proof tests [3]. This is an intriguing phenomenon associated with toroidal shell element, and this investigation studied the effect of external loads on the buckling of torispherical shells in the respect that stresses in one principal direction may be beyond the yield limit while the stresses in the other direction are lower than yield.

In this investigation, an analytical study of a torispherical shell was undertaken which had concentrated external loads applied to the apex (concentrated moment, axial load). The case of a toroidal shell segment under nonsymmetrical loading had to be solved before the analysis of the torispherical shell. After the solution for a toroidal shell was derived, a discontinuity analysis was made between a spherical, a toroidal, and cylindrical shell elements to arrive at a solution for torispherical shells.

1.1. *Previous contributions*

The most significant contributions to the solutions of toroidal shells have been made by Clark [4–7], Reissner [5–6], and Novozhilov [8]. In particular, an approximate solution for the axisymmetrical case has been derived by all three authors. Clark and Reissner derived their solution together, whereas Novozhilov's solution was developed independently.

The basic differential equations derived and solved by Clark were based on an axisymmetrically loaded shell. The loading considered was internal pressure. There were two basic equations in Clark's work; one equation was derived from two of the equations by substituting the kinematic relations, moment–curvature relations, and the force–displacement relations into them combining them, the other equation was derived in a similar manner except the two equilibrium equations and the stress-strain relations were substituted into a compatibility equation.

Clark used the horizontal and vertical directions as his principal axes and derived his equilibrium equations based on these axes.

The solution method used by Clark was to add these two equations together to form a complex functional differential equation. Then using a homogeneous solution for this type differential equation developed by Langer [9], the homogeneous solution was easily written down based upon certain assumptions made in equation in order to get it into the basic form derived by Langer. The main assumptions made were that the ratios t/r_0 and t/R_0 were very small relative to unity, say

$$t/r_0 \leqslant 1/20 \ ,$$

which, in most cases, falls within thin-shell theory. Also all terms in the equation which were multiplied by r_0/R_0 were considered small when compared with the terms multiplied by the ratio

$$\sqrt{12(1-\nu^2)}\, r_0^2/R_0 t \ ,$$

and therefore these terms were neglected. To get an approximate particular solution to his equation, Clark assumed that all terms in the equation multiplied $(r_0/R_0)^2$ or $\nu(r_0/R_0)$ were small and could be neglected. Because of these assumptions, this solution does not clearly show the interaction between the membrane and bending effects and therefore this solution must be used with caution.

Novozhilov derived his solution by the use of a general partial differential equation for a general shell of revolution and then reduced the equation for the special case of a toroidal segment. Novozhilov's method consists principally of replacing the deformation and change of curvature relations in the compatibility equations with moment–curvature and force–displacement relations and rewritting the equations in terms of forces and moments. Then these equations supplement the equilibrium equations for a general solution. By the introduction of moment and force complex auxiliary functions, the compatibility and equilibrium equations are added together by simple operations to form three first-order complex partial differential equations in three unknown complex stress resultants. These equations are then combined and reduced to form two second-order complex differential equations. By such an analytical technique, a whole series of complex differential equations can be derived. This methods is of general importance and applies to any type of loading which can be expressed by a Fourier expansion. Such an expansion will lead to a set of differential for each Fourier harmonic.

These general equations were reduced to the axisymmetric case, internal pressure, for a toroidal shell segment by Novozhilov. This equation was solved by using the method developed by Langer for the homogeneous solution [9]. Novozhilov derived a trigonometric series for the particular solution. This form of the solution is very difficult to use because of the many terms of the series that have to be used to get an accurate answer. In Novozhilov's book [8], graphs are drawn for the first six constants in the trigonometric series.

The two solutions discussed above are the only analytical solutions found in the literature. There have been many attempts to numerically integrate the equations for toroidal shell segments and some of the methods seems to be promising, notably, the work of Kalnins [10] for the numerical integration of shell equations. Steele's [11] dissertation develops numerical methods for the higher harmonics for the toroidal shell segments and gives a discussion of the assumptions made for this type of integrations.

In the analysis of pressure vessels, the first two Fourier harmonic solutions (0, 1) are the most important because the zero-order solutions are for the internal pressure and axial thrust loadings while the first-order solutions are for the concentrated moment loadings. By having analytical solutions for all shell elements, a discontinuity analysis of almost any shape vessel can be done with very little expenditure of time by use of a computer.

1.2. *Method of attack*

The torispherical shell with a nozzle at the apex and attached to a cylindrical shell was studied by a discontinuity analysis. In essence, the discontinuity analysis was between four shell segments: a spherical, a toroidal, and two cylindrical. Analytical solutions for the cylindrical and spherical shell elements were found in the literature [12–15]. These solutions have been extensively studied and compared against test results and have been found to be accurate [16–20]. The analytical solution for the toroidal shell was derived by using the general differential equations for a shell of revolution derived by Novozhilov and solving them for the first Fourier harmonic solution for a toroidal shell.

When these equations were combined and rearranged, they became one second-order complex differential equation. This single differential equation is exactly the same as Novozhilov's equation for the axisymmetric case except for one term. This particular term is

$$\frac{\alpha(1-\alpha \sin \phi)}{(1+\alpha \sin \phi) \sin^2 \phi} \tilde{N}_\phi ,$$

which is small when compared with the other $\tilde{N}_\phi$ term in the equation. The other term is

$$\sqrt{12(1-\nu^2)}\,(\alpha r_0/t)\,\tilde{N}_\phi .$$

Since α is small, the first term is much smaller than the second term and therefore it may be assumed that this term can be neglected without too much error. Then the equation reduces to the same form as Novozhilov's equation. The homogeneous solution for this equation was found by using the method develped by Langer [9]. The particular solution for this loading case and the axisymmetric case was derived by the variation of parameters method [21, 22]. The integration was done numerically on a digital computer.

2. Membrane solutions

In a membrane state of stress, a shell is assumed to have no bending loads or transverse shear loads applied to it. The membrane state is justified only when the shell has a very small bending stiffness or when the changes of curvature or twisting of the middle surface are very small. The membrane forces at a point of a shell represent a plane stress system in a plane tangent to the middle surface of a shell.

Consider a shell of revolution bounded by one or two parallel circles. If the surface loading components are arbitrary functions of the angles ϕ and θ, they may always be represented in the form [23]

$$P_\phi = \sum_0^\infty P_{\phi m} \cos m\theta + \sum_1^\infty \bar{P}_{\phi m} \sin m\theta ,$$

$$P_\theta = \sum_0^\infty P_{\theta m} \sin m\theta + \sum_1^\infty \bar{P}_{\theta m} \cos m\theta , \tag{1}$$

$$P_n = \sum_0^\infty P_{nm} \cos m\theta + \sum_1^\infty \bar{P}_{nm} \sin m\theta ,$$

where $P_{\phi m}, \ldots, \bar{P}_{nm}$ are functions of ϕ only. The first of the sums in eq. (1) represents that part of the load which is symmetric with respect to the plane of the meridian $\theta = 0$ and the second represents the antisymmetric terms.

To find a solution to the membrane state, pick one set of the terms, say:

$$P_\phi = P_{\theta m} \cos m\theta , \qquad P_\theta = P_{\theta m} \sin m\theta , \qquad P_n = P_{nm} \cos m\theta , \tag{2}$$

and find the particular case for these terms. Once this particular solution has been found, solutions for any distributed loading can be found by using the series [eq. (1)].

For this form of the surface loadings, the solution for the stress resultant may be written as follows:

$$N_\phi = N_{\theta m} \cos m\theta \,, \qquad N_\theta = N_{\theta m} \cos m\theta \,, \qquad N_{\phi\theta} = N_{\phi\theta m} \sin m\theta \,, \tag{3}$$

where $N_{\phi m}$, $N_{\theta m}$, $N_{\phi\theta m}$ are also functions of ϕ only. The positive directions of the stress resultants are shown in fig. 2.

Three membrane cases were considered in this study: internal pressure, an axial force applied around the upper edge of a torus, and an edge moment applied around the upper edge of a torus. The first two loading conditions are axisymmetrical about the axis of the shell while the last loading condition is nonsymmetrical about the axis. The axisymmetric loading cases are particular cases where the loading on the shell does not depend upon the circumferential angle, θ. This implies that the deformations of the shell are independent of this angle; they are symmetrical about the axis of the shell. These cases are obtained by omitting all but the first term ($m = 0$) in the series [eq. (1)]. This results in the deformation, stress resultants, and loadings being functions of ϕ only. For these loading conditions, the stress resultants of membrane theory can be calculated by the well-known equations [24]:

$$N_\phi = \frac{C_1}{R_2 \sin^2 \phi} + \frac{1}{R_2 \sin^2 \phi} \int_{\phi_0}^{\phi} (P_n \cos \phi - P_\phi \sin \phi) R_1 R_2 \sin \phi \, d\phi \,, \tag{4}$$

$$N_\theta = R_2 P_n - \frac{C_1}{R_1 \sin^2 \phi} - \frac{1}{R_1 \sin^2 \phi} \int_{\phi_0}^{\phi} (P_n \cos \phi - P_\phi \sin \phi) R_1 R_2 \sin \phi \, d\phi \,, \tag{5}$$

where C_1 is an unknown constant that can be obtained by satisfying the boundary conditions.

In the nonaxisymmetrical case, the deformations and the stress resultants are not independent of the angle θ; but are assumed to be dependent on the first Fourier harmonic ($m = 1$) as given in the series [eq. (1)]. In particular, the interest is in the external moment applied to the shell at the upper edge, ϕ_0, which is symmetric with respect to the plane of the meridian $\theta = 0$ and varies as $\cos\theta$ in the circumferential direction [14]. Also in this par-

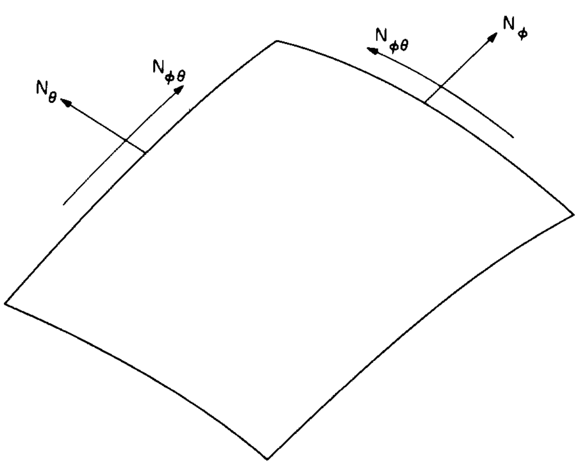

Fig. 2. The positive directions of the membrane forces acting on the shell element.

ticular case the surface loadings are assumed to be zero. This results in the deformations and stress resultants being functions of ϕ and the first harmonic of θ. The stress resultants of membrane theory for this case can be calculated by the equations [25]:

$$N_\phi = \frac{1}{R_2^2 \sin^3 \phi}(C_2 + C_1 \int_{\phi_0}^{\phi} R_1 \sin \phi \, d\phi + \int_{\phi_0}^{\phi} \Phi R_1 \sin \phi \, d\phi) \cos \theta , \tag{6}$$

$$N_\theta = [R_2 P_n - \frac{1}{R_1 R_2 \sin^3 \phi}(C_2 + C_1 \int_{\phi_0}^{\phi} R_1 \sin \phi \, d\phi + \int_{\phi_0}^{\phi} \Phi R_1 \sin \phi \, d\phi)] \cos \theta , \tag{7}$$

$$N_{\phi\theta} = \frac{\cot \phi}{(R_2 \sin \phi)^2}[C_2 - C_1 (R_2 \frac{\sin^2 \phi}{\cos \phi} - \int_{\phi_0}^{\phi} R_1 \sin \phi \, d\phi)$$

$$- \Phi \frac{R_2 \sin^2 \phi}{\cos \phi} + (P_n \cos \phi - P_\phi \sin \phi) \frac{R_2^3 \sin^3 \phi}{\cos \phi} + \int_{\phi_0}^{\phi} \Phi R_1 \sin \phi \, d\phi] \sin \theta , \tag{8}$$

where

$$\Phi = (P_n \cos \phi - P_\phi \sin \phi) R_2^2 \sin \phi - \int_{\phi_0}^{\phi} (P_n \sin \phi + P_\phi \cos \phi - P_\theta) \sin \phi \, R_1 R_2 \, d\phi ,$$

and where C_1 and C_2 are unknown constants that can be solved by satisfying the boundary conditions.

2.1. *Axial thrust load*

For the torus, the two radii in shell theory are given by

$$R_1 = r_o, \qquad R_2 = R_o (1 + \alpha \sin \phi) / \sin \phi , \tag{9}$$

where

$$\alpha = r_o / R_o .$$

Fig. 1 shows a meridional section of a toroidal shell and fig. 3 shows the distributed axial force, N_X, acting in the positive direction on the upper edge of the shell. The surface loadings are zero in this case and upon substitution of eq. (9) into eq. (4), eq. (4) reduces to

$$N_\phi = \frac{C_1}{R_o (1 + \alpha \sin \phi) \sin \phi} . \tag{10}$$

At the upper edge of the shell, $\phi = \phi_0$, the meridional stress resultant is given by

$$N_\phi = N_X / \sin \phi_0 . \tag{11}$$

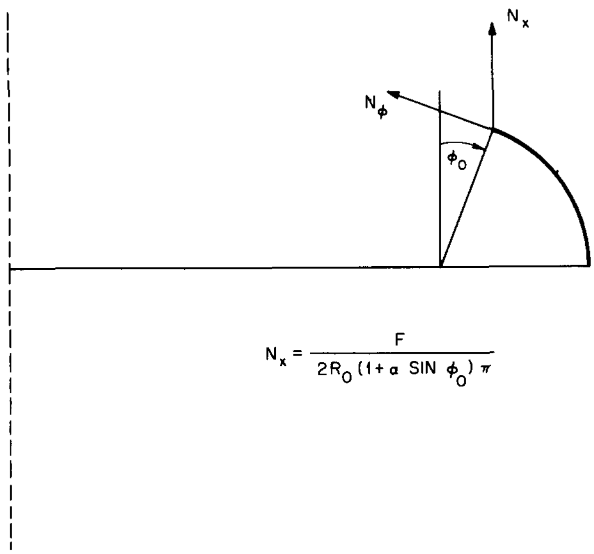

Fig. 3. The positive direction for the distributed axial force.

Solving eq. (10) for C_1 at $\phi = \phi_0$ by using eq. (11) and substituting C_1 back into eq. (10) yields

$$N_\phi = \frac{(1 + \alpha \sin \phi_0) N_X}{(1 + \alpha \sin \phi) \sin \phi} \tag{12}$$

for the distributed axial load, N_X.

The circumferential stress resultant for the axial thrust case can be found by substituting the value of C_1 into eq. (5), which becomes

$$N_\theta = -\frac{(1 + \alpha \sin \phi_0) N_X}{\alpha \sin^2 \phi} . \tag{13}$$

2.2. *Internal pressure*

The surface loading components for internal pressure loading are given by

$$P_n = P, \qquad P_\phi = 0, \qquad P_\theta = 0 . \tag{14}$$

Eqs. (4) and (5) by using eqs. (14) and (9) can be written as

$$N_\phi = \frac{C_1}{R_0 (1 + \alpha \sin \phi) \sin \phi} + \frac{P r_0}{(1 + \alpha \sin \phi) \sin \phi} \int_{\phi_0}^{\phi} \cos \phi \, (1 + \alpha \sin \phi) \, \mathrm{d}\phi , \tag{15}$$

$$N_\theta = \frac{P R_0 (1 + \alpha \sin \phi)}{\sin \phi} - \frac{C_1}{r_0 \sin^2 \phi} - \frac{P R_0}{\sin^2 \phi} \int_{\phi_0}^{\phi} \cos \phi \, (1 + \alpha \sin \phi) \, \mathrm{d}\phi . \tag{16}$$

If at the upper edge of the shell, $\phi = \phi_o$, the meridional stress resultant is given by N_ϕ^*, then eq. (15) becomes

$$N_\phi = \frac{(1+\alpha \sin \phi_o) \sin \phi_o}{(1+\alpha \sin \phi) \sin \phi} N_\phi^* + \frac{P r_o}{(1+\alpha \sin \phi) \sin \phi} [\sin \phi + \tfrac{1}{2}\alpha \sin^2 \phi - \sin \phi_o - \tfrac{1}{2}\alpha \sin^2 \phi_o] \; . \tag{17}$$

The internal pressure causes an axial force to act at the upper edge of the shell, $\phi = \phi_o$, which gives the stress resultant N_ϕ^*. The distributed axial force acting at this edge is given by

$$N_X = \frac{(R_2 \sin \phi_o)^2 \pi P}{2\pi R_2 \sin \phi_o} = \frac{R_o (1+\alpha \sin \phi_o) P}{2} \; . \tag{18}$$

Substituting this expression into eq. (12) at $\phi = \phi_o$ and in turn substituting this relation into eq. (17) for N_ϕ^* leads to the equation for the meridional stress resultant under internal pressure which is

$$N_\phi = \frac{r_o P}{2} \left[\frac{1+\alpha \sin \phi}{\alpha \sin \phi} \right] . \tag{19}$$

Then from eq. (16) the circumferential stress resultant is

$$N_\theta = \tfrac{1}{2} r_o P \, [1 - 1/(\alpha \sin \phi)^2] \; . \tag{20}$$

2.3. *Edge moment loading*

In this loading case, the external moment applied at the upper edge, $\phi = \phi_o$, varies as $\cos \theta$ about the meridional plane $\theta = 0$ and the surface loads are equal to zero. Eqs. (6), (7), and (8) then reduce to

$$N_\phi = \frac{1}{R_o^2 (1+\alpha \sin \phi)^2 \sin \phi} [C_2 + C_1 \int_{\phi_o}^{\phi} r_o \sin \phi \, d\phi] \cos \theta \; , \tag{21}$$

$$N_\theta = -\frac{1}{r_o R_o (1+\alpha \sin \phi) \sin^2 \phi} [C_2 + C_1 \int_{\phi_o}^{\phi} r_o \sin \phi \, d\phi] \cos \theta \; , \tag{22}$$

$$N_{\phi\theta} = \frac{\cot \phi}{R_o^2 (1+\alpha \sin \phi)^2} \left[C_2 + C_1 \left(\frac{R_o (1+\alpha \sin \phi)}{\cot \phi} - \int_{\phi_o}^{\phi} r_o \sin \phi \, d\phi \right) \right] \sin \theta \; . \tag{23}$$

The two constants C_1 and C_2 in the expressions must be determined from the given values of the forces N_ϕ and $N_{\phi\theta}$ at the upper edge of the shell.

Since the ultimate objective of this study is to obtain a solution for the torispherical shell, the membrane forces acting on the upper edge of the torus will be the membrane forces from the spherical portion of the shell. For a concentrated moment acting at the apex of the spherical shell, the membrane forces acting at the lower edge of the spherical shell which are the same membrane forces acting at the upper edge of the torus, are given by [14]

$$N_\phi^S = \frac{M \cos \theta}{\pi R_S^2 \sin^3 \phi_o}, \qquad N_{\phi\theta}^S = \frac{M \cos \phi_o \sin \theta}{\pi R_S^2 \sin^2 \phi_o} . \tag{24}$$

The radius of the sphere is denoted by R_S. The relationship between R_S and the geometry of the torus is given by the expression

$$R_S \sin\phi_o = R_o (1 + \alpha \sin\phi_o)\,. \tag{25}$$

Rewritting eq. (24) using the above relation gives

$$N_\phi^S = \frac{M\cos\theta}{\pi R_o^2 (1+\alpha\sin\phi_o)^2 \sin\phi_o}\,, \qquad N_{\phi\theta}^S = \frac{M\cos\phi_o \sin\theta}{\pi R_o^2 (1+\alpha\sin\phi_o)^2 \sin\phi_o}\,. \tag{26}$$

Evaluating the eqs. (21) and (23) at $\phi = \phi_o$ requires that the unknown constant C_1 be set equal to zero since the evaluated integrals are zero and C_1 would be indefinite. By setting the right-hand sides of the evaluated equations equal to the right-hand side of eq. (26), C_2 was found and its value is

$$C_2 = M/\pi\,. \tag{27}$$

Then eqs. (21), (22), and (23) can be rewritten as:

$$N_\phi = \frac{M\cos\theta}{\pi R_o^2 (1+\alpha\sin\phi)^2 \sin\phi}\,, \qquad N_\theta = \frac{M\cos\theta}{\pi r_o R_o (1+\alpha\sin\phi)\sin^2\phi}\,, \qquad N_{\phi\theta} = \frac{M\cos\phi\sin\theta}{\pi R_o^2 (1+\alpha\sin\phi)^2 \sin\phi}\,. \tag{28}$$

2.4. *Secondary membrane resultants*

When discussing the membrane stress resultants in the first part of this section, the assumption was made that all bending and twisting moments were zero. Now a real shell of thickness, t, has a finite bending rigidity:

$$K = \frac{Et^3}{12(1-\nu^2)}\,. \tag{29}$$

This indicates that all shells even though loaded only in a membrane state will have some changes of curvature or some twisting of their middle surfaces. Examination of the compatibility equations shows the relationship between the in-plane stress resultants and the bending or twisting stress resultants. The compatibility conditions as derived by Novozhilov [26] are:

$$\frac{\partial}{\partial\phi}[R_2 \sin\phi\,(M_\theta - \nu M_\phi)] - (1+\nu)\left(R_1 \frac{\partial M_{\phi\theta}}{\partial\theta}\right) - \frac{\partial R_2\sin\phi}{\partial\phi}(M_\phi - \nu M_\phi)$$

$$= \frac{t^2}{12R_1}\left\{\frac{\partial}{\partial\phi}[R_2\sin\phi\,(N_\theta - \nu N_\phi)] - \frac{\partial R_2\sin\phi}{\partial\phi}(N_\phi - \nu N_\theta) - 2(1+\nu)\frac{\partial(R_1 N_{\phi\theta})}{\partial\theta}\right\}\,, \tag{30}$$

$$R_1 \frac{\partial}{\partial\theta}(M_\phi - \nu M_\theta) - (1+\nu)\;2\left(\frac{\partial R_2\sin\phi}{\partial\phi} M_{\phi\theta} + R_2\sin\phi\frac{\partial M_{\phi\theta}}{\partial\phi}\right)$$

$$= \frac{t^2}{12R_2}\left\{R_1\frac{\partial}{\partial\theta}(N_\phi - \nu N_\theta) - 2(1+\nu)\frac{\partial}{\partial\phi}(R_2\sin\phi\,N_{\phi\theta}) - 2(1+\nu)\frac{R_2}{R_1}\,\frac{\partial R_2\sin\phi}{\partial\phi}N_{\phi\theta}\right\}\,, \tag{31}$$

$$\frac{M_\theta - \nu M_\phi}{R_1} + \frac{M_\phi - \nu M_\theta}{R_2} = -\frac{t^2}{12}\,\frac{1}{R_1 R_2 \sin\phi}\left\{\frac{\partial}{\partial\phi}\,\frac{1}{R_1}\left[R_2 \sin\phi \frac{\partial}{\partial\phi}[N_\theta - \nu N_\phi] - (1+\nu)\left(R_1 \frac{\partial N_{\phi\theta}}{\partial\theta}\right)\right]\right.$$

$$\left. + \frac{\partial}{\partial\theta}\,\frac{1}{R_2 \sin\phi}\left[R_1 \frac{\partial}{\partial\theta}(N_\phi - \nu N_\theta) - (1+\nu)\left(\frac{\partial (R_2 \sin\phi\, N_{\phi\theta})}{\partial\phi}\right) + \frac{\partial R_2 \sin\phi}{\partial\phi} N_{\phi\theta}\right]\right\} . \tag{32}$$

For the axisymmetric case, the equations can be rewritten in the following form:

$$\frac{\partial}{\partial\phi}[R_2 \sin\phi (M_\theta - \nu M_\phi)] - \frac{\partial R_2 \sin\phi}{\partial\phi}(M_\phi - \nu M_\theta)$$

$$= \frac{t^2}{12R_1}\left\{\frac{\partial}{\partial\phi}[R_2 \sin\phi (N_\theta - \nu N_\phi)] - \frac{\partial R_2 \sin\phi}{\partial\phi}(N_\phi - \nu N_\theta)\right\} , \tag{33}$$

$$\frac{M_\theta - \nu M_\phi}{R_1} + \frac{M_\phi - \nu M_\theta}{R_2} \approx 0 . \tag{34}$$

The last equation is based upon the assumption that $t^2/12R_1^2 R_2$ or $t^2/12R_1 R_2^2$ is much smaller than $t^2/12R_1$ and therefore the right-hand side of eq. (32) is neglected. For the toroidal shell, the following approximate relations for the secondary stress resultants for axisymmetric loadings are derived from eqs. (33) and (34):

$$M_\phi = \frac{t^2}{12r_0}(N_\phi - \nu N_\theta) , \qquad M_\theta = -\frac{t^2}{12R_0}\,\frac{\sin\phi}{1+\alpha\sin\phi}(N_\phi - \nu N_\theta) . \tag{35}$$

These equations are based on the assumptions that terms multiplied by $\nu/R_1, \nu/R_2, \nu R_1/R_2$ can be neglected.

For the nonaxisymmetric case, particularly the externally applied moment loading, eqs. (30), (31), and (32) can be written in the following form:

$$\frac{\partial}{\partial\phi}[R_2 \sin\phi (M_\theta - \nu M\phi)] - (1+\nu) R_1 \frac{\partial M_{\phi\theta}}{\partial\theta} - \frac{\partial R_2 \sin\phi}{\partial\phi}(M_\phi - \nu M_\theta)$$

$$= \frac{t^2}{12R_1}\left\{\frac{\partial}{\partial\phi}[R_2 \sin\phi (N_\theta - \nu N_\phi)] - \frac{\partial R_2 \sin\phi}{\partial\phi}(N_\phi - \nu N_\theta) - 2(1+\nu) R_1 \frac{\partial N_{\phi\theta}}{\partial\theta}\right\} , \tag{36}$$

$$R_1 \frac{\partial}{\partial\theta}(M_\phi - \nu M_\theta) - (1+\nu)\left(2R_1 \cos\phi M_{\phi\theta} + R_2 \sin\phi \frac{\partial M_{\phi\theta}}{\partial\phi}\right)$$

$$= \frac{t^2}{12R_2}\left\{R_1 \frac{\partial}{\partial\theta}(N_\phi - \nu N_\theta) - 2(1+\nu)\left(R_1 \cos\phi N_{\phi\theta} + R_2 \sin\phi \frac{\partial N_{\phi\theta}}{\partial\phi}\right) - 2(1+\nu) R_2 \cos\phi N_{\phi\theta}\right\} , \tag{37}$$

$$\frac{M_\phi - \nu M_\theta}{R_1} + \frac{M_\phi - \nu M_\theta}{R_2} \approx 0 \,. \tag{38}$$

For the toroidal shell subjected to nonaxisymmetric loadings, the following approximate relations for the secondary membrane resultants are derived from eqs. (36) and (38):

$$M_\phi = \frac{t^2}{12 r_0}(N_\phi - \nu N_\theta)\,, \qquad M_\theta = -\frac{t^2}{12 R_0}\,\frac{\sin\phi}{1+\alpha\sin\phi}(N_\phi - \nu N_\theta)\,, \qquad M_{\phi\theta} = \frac{t^2}{12 r_0} N_{\phi\theta} \,. \tag{39}$$

The equations were derived by using only eqs. (36) and (38). Since there are differences between eqs. (36) and (37), the derivation using eq. (37) would lead to different but smaller values for the moments. Therefore, since eqs. (39) give larger values for the moments and agrees very well with the forms for the moments used in ref. [14], these relations for the moments will be used. Note also that the multiplier 2 was dropped in front of the last term in eq. (36) for the derivation of eqs. (39) because of the assumptions $M_{\phi\theta} = M_{\theta\phi}$ and $N_{\phi\theta} = N_{\theta\phi}$.

3. Bending solution

Novozhilov reduced the analyses of shells of revolution to two second-order complex differential equations [27]:

$$G_m(\widetilde{U}_m) + m^2\left[1 - \mathrm{i}C\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\phi}\right]\widetilde{N}_m = F_m(\phi)\,, \tag{40}$$

$$-\mathrm{i}C G_m(\widetilde{N}_m) + \widetilde{N}_m + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\phi}\,\widetilde{U}_m = R_2 P_{nm}\,, \tag{41}$$

where

$$F_m(\phi) = \frac{1}{R_1 R_2 \sin\phi}\left\{\frac{\mathrm{d}}{\mathrm{d}\phi}\left[(P_{nm}\cos\phi - P_{\phi m}\sin\phi) R_2^3 \sin^2\phi\right] + m P_{\phi m} R_1 R_2^2 \sin^2\phi\right\} . \tag{42}$$

$$G(\ldots) = \frac{1}{R_1 R_2 \sin\phi}\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{R_2^2\sin\phi}{R_1}\,\frac{\mathrm{d}(\ldots)}{\mathrm{d}\phi}\right] - \frac{m^2}{R_2\sin^2\phi}(\ldots)\,, \tag{43}$$

$$\widetilde{U}_m = \widetilde{N}_{\phi m} R_2 \sin^2\phi - \mathrm{i}C\frac{R_2}{R_1}\sin\phi\cos\phi\,\frac{\mathrm{d}N_m}{\mathrm{d}\phi}\,, \tag{44}$$

$$\frac{\mathrm{d}U_n}{\mathrm{d}\phi} + R_1\sin\phi\,\frac{\mathrm{d}N_{\phi\theta m}}{\mathrm{d}\theta} - \mathrm{i}C\frac{R_1}{R_2}\cot\phi\,\frac{\mathrm{d}^2\widetilde{N}}{\mathrm{d}\theta^2} = (P_{nm}\cos\phi - P_{\phi m}\sin\phi) R_1 R_2 \sin\phi\,, \tag{45}$$

$$C = \frac{t}{\sqrt{12(1-\nu^2)}}\,, \tag{46}$$

$$\widetilde{N}_{\phi m} = N_{\phi m} - \frac{\mathrm{i}}{C}\,\frac{M_{\theta m} - \nu M_{\phi m}}{1 - \nu^2}\,, \tag{47}$$

$$\widetilde{N}_{\theta m} = N_{\theta m} - \frac{\mathrm{i}}{C}\,\frac{M_{\phi m} - \nu M_{\theta m}}{1 - \nu^2}\,, \tag{48}$$

$$\widetilde{N}_{\phi\theta m} = N_{\phi\theta m} + \frac{\mathrm{i}}{C}\,\frac{M_{\phi\theta m}}{1 - \nu}\,, \tag{49}$$

$$\widetilde{N}_m = \widetilde{N}_{\phi m} + \widetilde{N}_{\theta m}\,, \tag{50}$$

$$\widetilde{N}_\phi = \widetilde{N}_{\phi m} \cos m\theta\,, \qquad \widetilde{N}_\phi = \widetilde{N}_{\phi m} \cos m\theta\,, \qquad \widetilde{N}_{\phi\theta} = \widetilde{N}_{\phi\theta m} \sin m\theta\,, \tag{51}$$

$$N_\phi = N_{\phi m} \cos m\theta\,, \qquad N_\theta = N_{\theta m} \cos m\theta\,, \qquad N_{\phi\theta} = N_{\phi\theta m} \sin m\theta\,, \tag{52}$$

$$M_\phi = M_{\phi m} \cos m\theta\,, \qquad M_\theta = M_{\theta m} \cos m\theta\,, \qquad M_{\phi\theta} = M_{\phi\theta m} \sin m\theta\,, \tag{53}$$

3.1. *Nonaxisymmetrical bending case ($m = 1$)*

The solution for the toroidal shell for the first Fourier harmonic is developed in the following manner. Eqs. (40) and (41) are written for the first harmonic and eq. (40) is subtracted from eq. (41) yielding:

$$G_1(\widetilde{W}) - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\phi}\,\widetilde{W} = F_1(\phi) - R_2 P_{n1}\,, \tag{54}$$

where

$$\widetilde{W} = \widetilde{U}_1 + \mathrm{i}C\widetilde{N}_1\,. \tag{55}$$

Upon integrating eq. (54), the expression for $\widetilde{W}$ is

$$\widetilde{W} = \frac{1}{R_2 \sin\phi}\left(C_1 + C_2 \int_{\phi_o}^{\phi} R_1 \sin\phi \,\mathrm{d}\phi + \int \Phi R_1 \sin\phi\,\mathrm{d}\phi\right)\,, \tag{56}$$

where

$$\Phi = (P_{n1}\cos\phi - P_{\phi 1}\sin\phi) R_2^2 \sin\phi - \int_{\phi_o}^{\phi} (P_{n1}\sin\phi + P_{\phi 1}\cos\phi - P_{\theta 1}) R_1 R_2 \sin\phi\,\mathrm{d}\phi\,,$$

and C_1 and C_2 are real arbitrary constants. The lower limit of integration ϕ_o is identified with the angle corresponding to the upper edge of the toroidal shell.

The relation between $\widetilde{U}_1$ and $\widetilde{N}_1$ can be established after substituting the expression for $\widetilde{W}$ into eq. (55). Introducing this new value of $\widetilde{U}_1$ into eq. (41), the following complex second-order differential equation is acquired:

$$G_1(N_1) + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\phi}\widetilde{N}_1 + \frac{\mathrm{i}}{C}\widetilde{N}_1 = \frac{\mathrm{i}}{C}F_1(\phi)\,, \tag{57}$$

where

$$F_1(\phi) = R_2P_{n1} - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{R_2\sin^3\phi}\left[C_1 + C_2\int_{\phi_0}^{\phi} R_1\sin\phi\,\mathrm{d}\phi + \int_{\phi_0}^{\phi}\Phi R_1\sin\phi\,\mathrm{d}\phi\right]\,. \tag{58}$$

By comparing eq. (58) with eqs. (6) and (7), it is readily seen that eq. (58) can be expressed by the relation:

$$F_1(\phi) = N_{\phi 1}^* + N_{\theta 1}^*\,, \tag{59}$$

where $N_{\phi 1}^*$ an $N_{\theta 1}^*$ are the membrane stress resultants.

The analysis of shells of revolution of arbitrary shape for the first Fourier harmonic has been reduced to the integration of a complex secondorder linear differential equation given by eq. (57). This differential equation was first derived by Novozhilov [27].

Expanding eq. (57) and using eq. (9), eq. (57) can be written as:

$$\frac{\mathrm{d}^2N_1}{\mathrm{d}\phi^2} + \left(\frac{\alpha\sin\phi - 1}{1 + \alpha\sin\phi}\right)\cot\phi\,\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\phi} + \frac{\alpha(1 - \alpha\sin\phi)}{(1 + \alpha\sin\phi)^2\sin\phi}\widetilde{N}_1 + \mathrm{i}2d^2\,\frac{\sin\phi}{1 + \alpha\sin\phi}\widetilde{N}_1 = \mathrm{i}2d^2\,\frac{\sin\phi}{1 + \alpha\sin\phi}(N_{\phi 1}^* + N_{\theta 1}^*)\,, \tag{60}$$

where

$$2d^2 = \sqrt{12(1 - \nu^2)}\,r_0^2/R_0t = r_0^2/R_0C\,.$$

Multiplying each term of the equation by $(1 + \alpha\sin\phi)/\sin\phi$ and simplifying leads to the differential equation

$$\frac{1}{1 + \alpha\sin\phi}\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{(1 + \alpha\sin\phi)^2}{\sin\phi}\,\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\phi}\right] + \frac{\alpha(1 - \alpha\sin\phi)}{(1 + \alpha\sin\phi)\sin^2\phi}\widetilde{N}_1 + \mathrm{i}2d^2\widetilde{N}_1 = \mathrm{i}2d^2(N_{\phi 1}^* + N_{\theta 1}^*)\,. \tag{61}$$

Upon inspection of eq. (61), it is readily seen that the term

$$\frac{\alpha(1 - \alpha\sin\phi)}{(1 + \alpha\sin\phi)\sin^2\phi}\widetilde{N}_1$$

is obviously small since α is small and can be neglected when compared with the term $\mathrm{i}2d^2\widetilde{N}_1$ since

$$\sqrt{12(1 - \nu^2)}\,\frac{\alpha r_0}{t} >> \frac{\alpha(1 - \alpha\sin\phi)}{(1 + \alpha\sin\phi)\sin^2\phi}\,.$$

Therefore, it is assumed that this term can be neglected and eq. (61) becomes:

$$\frac{1}{1+\alpha \sin \phi} \frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{(1+\alpha \sin \phi)^2}{\sin \phi} \frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\phi}\right] + \mathrm{i}2d^2\widetilde{N}_1 = \mathrm{i}2d^2\,(N^*_{\phi 1} + N^*_{\theta 1})\,. \tag{62}$$

The homogeneous solution for this equation was found by using a method developed by Langer [9]. The particular solution to this equation is considered later. The homogeneous solution is:

$$\frac{\mathrm{d}\widetilde{N}_1}{\mathrm{d}\phi} = \frac{\sin \phi}{(1+\alpha \sin \phi)^2}\widetilde{Z}\,, \tag{63}$$

where

$$\widetilde{Z} = \eta\,[\widetilde{C}_1\widetilde{Z}_1\,(\sqrt{2\mathrm{i}}\epsilon) + \widetilde{C}_2\widetilde{Z}_2\,(\sqrt{2\mathrm{i}}\epsilon)]\;,$$

$$\eta = \frac{(1+\alpha \sin \phi)^{\frac{3}{4}}}{(\sin \phi)^{\frac{1}{4}}}\left(\frac{\epsilon}{d}\right)^{\frac{1}{6}}\,,$$

$$\widetilde{Z}_1\,(\sqrt{2\mathrm{i}}\epsilon) = (\sqrt{2\mathrm{i}}\epsilon)^{\frac{1}{3}} H^{(1)}_{\frac{1}{3}}\,(\sqrt{2\mathrm{i}}\epsilon)\,, \qquad \widetilde{Z}_2\,(\sqrt{2\mathrm{i}}\epsilon) = (\sqrt{2\mathrm{i}}\epsilon)^{\frac{1}{3}} H^{(2)}_{\frac{1}{2}}\,(\sqrt{2\mathrm{i}}\epsilon)\,,$$

$$\epsilon = \frac{2}{3}d\phi^{\frac{3}{2}}\left[1 - \frac{1}{28}\phi^2 + \frac{1}{5280}\phi^4 - \ldots \quad \frac{3}{10}\alpha\,\phi\left(1 - \frac{5}{36}\phi^2 + \frac{11}{1248}\phi^4 - \ldots\right)\right.$$
$$\left. + \frac{9}{56}\alpha^2\,\phi^2\left(1 - \frac{35}{135}\phi^2 + \frac{49}{1440}\phi^4 - \ldots\right)\right]\,.$$

Eqs. (47), (48), and (49) were solved for the stress resultants using the solution to differential equation (62) and they are as follows:

$$N_\phi = \left\{-\frac{\alpha}{2d^2}\left[\frac{\cos \phi}{(1+\alpha \sin)^2}\,\mathrm{Im}\,(\widetilde{Z})\right] + \frac{(t/r_\mathrm{o})^2}{12(1-\nu^2)}\,\frac{1}{(1+\alpha \sin \phi)^2 \sin \phi}\mathrm{Re}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\right\}\cos \theta\,,$$

$$N_\theta \left\{-\frac{1}{2d^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{1}{(1+\alpha \sin \phi)}\,\mathrm{Im}\,(\widetilde{Z})\right] + \frac{(t/r_\mathrm{o})^2}{12(1-\nu^2)}\,\frac{\cot \theta}{(1+\alpha \sin \phi)^2}\,\mathrm{Re}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\right\}\cos \phi\,,$$

$$N_{\phi\theta} = \left\{-\frac{\alpha}{2d^2}\,\frac{1}{(1+\alpha \sin)^2}\,\mathrm{Im}\,(\widetilde{Z}) + \frac{(t/r_\mathrm{o})^2}{12(1-\nu^2)}\,\frac{\cot \phi}{(1+\alpha \sin \phi)^2}\,\mathrm{Re}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\right\}\sin \phi\,, \tag{64}$$

$$M_\phi = \left\{-\frac{C}{2d^2}\left[\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{1+\alpha \sin \phi}\,\mathrm{Re}\,(\widetilde{Z})\right) + \frac{\nu\,\alpha \cos \phi}{(1+\alpha \sin \phi)^2}\,\mathrm{Re}\,(\widetilde{Z})\right]\right.$$

$$+\frac{C(t/r_o)^2}{12(1+\nu)}\frac{1}{(1+\alpha\sin\phi)^2\sin\phi}\operatorname{Im}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\Bigg\}\cos\theta\ ,$$

$$M_\theta=\Bigg\{-\frac{C}{2d^2}\left[\frac{\alpha\cos\phi}{(1+\alpha\sin\phi)^2}\operatorname{Re}(\widetilde{Z})+\nu\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{1+\alpha\sin\phi}\operatorname{Re}(\widetilde{Z})\right)\right]$$

$$-\frac{C(t/r_o)^2}{12(1+\nu)}\frac{1}{(1+\alpha\sin\phi)^2\sin\phi}\operatorname{Im}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\Bigg\}\cos\theta\ ,$$

$$M_{\phi\theta}=\left\{\frac{C(1-\nu)}{(1+\alpha\sin\phi)^2}\left[\frac{\alpha}{2d^2}\operatorname{Re}(\widetilde{Z})+\frac{(t/r_o)^2}{12(1-\nu^2)}\cot\phi\operatorname{Im}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\right]\right\}\sin\theta\ .$$

$$Q_\phi=-\frac{C}{r_o}\frac{\sin\phi}{(1+\alpha\sin\phi)^2}\operatorname{Im}(\widetilde{Z})\cos\theta\ ,\qquad Q_\theta=\frac{C}{2d^2}\frac{1}{R_o(1+\alpha\sin\phi)^2}\operatorname{Re}\left(\frac{\mathrm{d}\widetilde{Z}}{\mathrm{d}\phi}\right)\sin\theta\ .$$

In order to do a discontinuity analysis of a toroidal shell attached to another shell, the slope or angle of rotation of the tangent to the meridian and the horizontal deflection of the shell are needed. The deflection was calculated by the well-known relation:

$$\delta=\frac{R_o(1+\alpha\sin\phi)}{Et}(N_\theta-\nu N_\phi) \tag{65}$$

and the slope in the meridional direction was derived by solving the following two equations simultaneously

$$M_\phi=\frac{Et^3}{12(1-\nu^2)}(k_\phi+\nu k_\theta)\ ,\qquad M_\theta=\frac{Et^3}{12(1-\nu^2)}(k_\theta+\nu k_\phi)\ , \tag{66}$$

where

$$k_\phi=\frac{1}{r_o}\frac{\mathrm{d}X_\phi}{\mathrm{d}\phi}\ ,\qquad k_\theta=\frac{1}{R_o(1+\alpha\sin\phi)}X_\theta+\frac{\cos\phi}{R_o(1+\alpha\sin\phi)}X_\phi\ . \tag{67}$$

The relation for the slope in the meridional direction is:

$$X_\phi=-\frac{1}{Et}\frac{1}{\alpha(1+\alpha\sin\phi)}\operatorname{Re}(\widetilde{Z})\ . \tag{68}$$

The positive directions of the stress resultants and surface loads are shown in fig. 4, while in fig. 5a cross section of the shell along a ϕ line is shown.

3.2. *Axisymmetrical bending case ($m = 0$)*

This is the particularly important case when $m = 0$ in eqs. (40) through (53); therefore, the surface loadings are

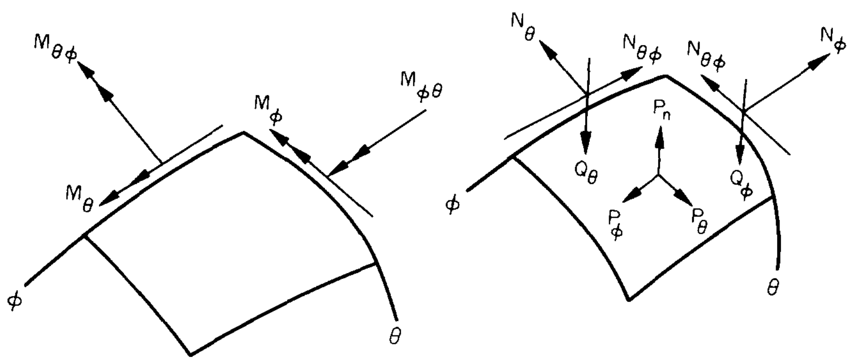

Fig. 4. Positive directions of the stress resultants and surface loads.

symmetrical with respect to the axis of the shell and the surface loading component P_θ is absent. In order for the deformations of the shell to be axisymmetric, the surface loadings and the edge loadings must be axisymmetric.

If the edge loadings and the surface loadings are symmetrical, then the loadings only produce the stress resultants $N_\phi, N_\theta, M_\phi, M_\theta, Q_\phi$. Therefore, the state of stress in the shell can be completely characterized by the complex functions $\widetilde{N}_\phi$ and $\widetilde{N}_\theta$ (as functions of ϕ only).

Under these conditions, eqs. (40) through (53) take the form

$$G(\widetilde{U}) = F_m(\phi)\,, \tag{69}$$

$$-\mathrm{i}CG(\widetilde{N}) + \widetilde{N} + \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{1}{\sin^2\phi}\,\widetilde{U} = R_2 P_n\,, \tag{70}$$

where

$$F_m(\phi) = \frac{1}{R_1 R_2 \sin\phi}\left\{\frac{\mathrm{d}}{\mathrm{d}\phi}\,[(P_n\cos\phi - P_\phi\sin\phi]\,R_2^3\sin^2\theta]\right\}\,, \tag{71}$$

$$G(\,\ldots\,) = \frac{1}{R_1 R_2\sin\phi}\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{R_2^2\sin\phi}{R_1}\,\frac{\mathrm{d}(\,\ldots\,)}{\mathrm{d}\phi}\right], \tag{72}$$

$$\widetilde{U} = \widetilde{N}_\phi R_2\sin^2\phi - \mathrm{i}C\frac{R_2}{R_1}\sin\phi\cos\phi\,\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}\phi}\,, \qquad \widetilde{N}_\phi = N_\phi - \frac{\mathrm{i}}{C}\,\frac{M_\theta - \nu M_\phi}{1-\nu^2}\,, \tag{73, 74}$$

$$\widetilde{N}_\theta = N_\theta - \frac{\mathrm{i}}{C}\,\frac{M_\phi - \nu M_\theta}{1-\nu^2}\,, \qquad \widetilde{N} = \widetilde{N}_\phi + \widetilde{N}_\theta\,, \qquad \frac{\mathrm{d}U}{\mathrm{d}\phi} = (P_n\cos\phi - P_\phi\sin\phi)\,R_1R_2\sin\phi\,. \tag{75–77}$$

Upon inspection of eqs. (69) and (77), it is seen that these two equations are the same. Integrating eq. (77) with respect to ϕ yields

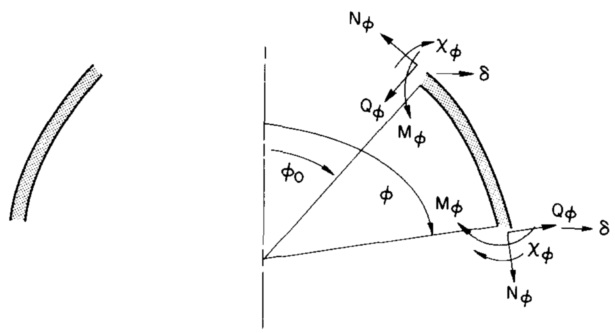

Fig. 5. The shell showing a cross section along a ϕ-coordinate line.

$$\widetilde{U} = C_1 + \int_{\phi_0}^{\phi} (P_n \cos\phi - P_\phi \sin\phi) R_1 R_2 \sin\phi \, d\phi \,, \tag{78}$$

where C_1 is a real constant and the lower limit ϕ_0 is identified with the angle corresponding to the upper edge of the toroidal shell.

Substituting $\widetilde{U}$ into eq. (70), the following complex second-order differential equation is acquired:

$$-\mathrm{i}CG(\widetilde{N}) + \widetilde{N} = F(\phi) \,, \tag{79}$$

where

$$F(\phi) = P_n R_2 - \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \frac{1}{\sin^2\phi} \left[C_1 + \int_{\phi_0}^{\phi} (P_n \cos\phi - P_\phi \sin\phi) R_1 R_2 \sin\phi \, d\phi\right] \,. \tag{80}$$

By comparing eq. (79) with eqs. (4) and (5), it is readily seen that eq. (80) can be expressed by the relation

$$F(\phi) = N_\phi^* + N_\theta^* \,, \tag{81}$$

where N_ϕ^* and N_θ^* are the membrane stress resultants.

Therefore the axisymmetric analysis of a shell of revolution of arbitrary shape has been reduced to the integration of a complex second-order linear differential equation given by eq. (79).

Eq. (79) was expanded and using eqs. (9) was rewritten as

$$\frac{d^2\widetilde{N}}{d\phi^2} + \left[\frac{\alpha \sin\phi - 1}{1 + \alpha \sin\phi}\right] \cot\phi \, \frac{d\widetilde{N}}{d\phi} + \mathrm{i}2d^2 \, \frac{\sin\phi}{1 + \alpha\sin\phi} \widetilde{N} = \mathrm{i}2d^2 \, \frac{\sin\phi}{1 + \alpha\sin\phi} (N_\phi^* + N_\theta^*) \,. \tag{82}$$

Multiplying each term of the equation by $(1 + \alpha \sin\phi)/\sin\phi$ and simplifying led to the relation

$$\frac{1}{1 + \alpha\sin\phi} \, \frac{d}{d\phi} \left[\frac{(1 + \alpha\sin\phi)^2}{\sin\phi} \, \frac{d\widetilde{N}}{d\phi}\right] + \mathrm{i}2d^2\widetilde{N} = \mathrm{i}2d^2 (N_\phi^* + N_\theta^*) \,. \tag{83}$$

The homogeneous solution for a similar differential equation [eq. (62)] was found earlier and by analogy the solution for this equation can be written as

$$\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}\phi} = \frac{\sin\phi}{(1+\alpha\sin\phi)^2}\,\widetilde{Z}\,, \tag{84}$$

where the function $\widetilde{Z}$ is the same as given by eq. (63). Having found a solution for the differential equation [eq. (82)], eqs. (74) and (76) were solved for the stress resultants and they are as follows:

$$N_\phi = -\frac{\alpha\cos\phi}{(1+\alpha\sin\phi)^2}\,\frac{1}{2d^2}\,\mathrm{Im}(\widetilde{Z})\,, \qquad N_\theta = -\frac{1}{2d^2}\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{1}{1+\alpha\sin\phi}\,\mathrm{Im}(\widetilde{Z})\right],$$

$$M_\phi = -\frac{C}{2d^2}\left[\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{1+\alpha\sin\phi}\,\mathrm{Re}(\widetilde{Z})\right) + \frac{\nu\alpha\cos\phi}{(1+\alpha\sin\phi)^2}\,\mathrm{Re}(\widetilde{Z})\right],$$

$$M_\theta = -\frac{C}{2d^2}\left[\frac{\alpha\cos\phi}{(1+\alpha\sin\phi)^2}\,\mathrm{Re}(\widetilde{Z}) + \nu\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{1+\alpha\sin\phi}\,\mathrm{Re}(\widetilde{Z})\right)\right], \qquad Q_\phi = -\frac{C}{r_0}\,\frac{\sin\phi}{(1+\alpha\sin\phi)^2}\,\mathrm{Im}(\widetilde{Z})\,. \tag{85}$$

The horizontal deflection of the shell is given by eq. (65) and the slope was found by simultaneously solving eqs. (66), where

$$k_\phi = \frac{1}{r_0}\,\frac{\mathrm{d}X_\phi}{\mathrm{d}\phi}\,, \qquad k_\theta = \frac{\cos\phi}{R_0(1+\alpha\sin\phi)}\,X_\phi\,. \tag{86}$$

The relation for the slope is:

$$X_\phi = -\frac{1}{Et}\,\frac{1}{\alpha(1+\alpha\sin\phi)}\,\mathrm{Re}(\widetilde{Z})\,. \tag{87}$$

3.3. *The particular solution*

Since both eq. (62) and (83) have the same form, the same method of attack was used for both cases. To use Langer's method [9] the equations (62) and (83) are differential with respect to ϕ and multiplied by $(1+\alpha\sin\phi)^2$ in order to reduce them to the form

$$(1+\alpha\sin\phi)^2\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left\{\frac{1}{1+\alpha\sin\phi}\,\frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{(1+\alpha\sin\phi)^2}{\sin\phi}\,\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}\phi}\right]\right\} + \mathrm{i}2d^2(1+\alpha\sin\phi)^2\,\frac{\mathrm{d}\widetilde{N}}{\mathrm{d}\phi} = \mathrm{i}2d^2(1+\alpha\sin\phi)^2\,\frac{\mathrm{d}}{\mathrm{d}\phi}(N_\phi^* + N_\theta^*)\,.$$

Langer developed the homogeneous solution for equations of this form and the method of variation of parameters was used to derive the particular solution. First, the function

$$\widetilde{f}(x) = \mathrm{i}2d^2(1+\alpha\sin\phi)^2\,\frac{\mathrm{d}}{\mathrm{d}\phi}(N_\phi^* + N_\theta^*) \tag{88}$$

must be determined. This function will depend upon the loading for each particular case since it depends upon the corresponding membrane solution. The $\widetilde{f}(x)$ is calculated for each case as follows:

Pressure:

$$\tilde{f}(x) = \mathrm{i}2d^2 \; \frac{r_o P}{2} \; \frac{\cos\phi(2-\alpha\sin\phi)\,(1+\alpha\sin\phi)^2}{\alpha^2\sin^3\phi} \;,$$

Axial force:

$$\tilde{f}(x) = \mathrm{i}2d^2 \; \frac{A\,(1+\alpha\sin\phi_o)\cos\phi}{\alpha\sin^2\phi} \; \frac{2+3\,\alpha\sin\phi}{\sin\phi} \;,$$

Concentrated moment:

$$\tilde{f}(x) = \mathrm{i}2d^2 \; \frac{2M\cos\phi}{\pi R_o r_o} \; \frac{1+2\,\alpha\sin\phi}{(1+2\sin\phi)\,(\sin\phi)^3} \;. \tag{89}$$

The variation of parameter methods [21, 22] permits the determination of a particular integral of an equation of the form:

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + P_1(x)\,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + P_{n-1}(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + P_n(x)y = f(x) \;, \tag{90}$$

where the general solution of the related homogeneous equation

$$\frac{\mathrm{d}y^n}{\mathrm{d}x^n} + P_1(x)\,\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + P_{n-1}(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + P_n(x)y = 0 \tag{91}$$

is known.

Let the general solution of eq. (91) be of the form:

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n \;, \tag{92}$$

in which the C_n's are arbitrary constants, and assume that a set of n functions $V_1(x), V_2(x), \ldots, V_n(x)$ can be chosen so that

$$y = V_1 y_1 + V_2 y_2 + \ldots + V_n y_n \;, \tag{93}$$

is a particular solution of eq. (90). By a standard technique, the functions $\mathrm{d}/\mathrm{d}x(V_1), \mathrm{d}/\mathrm{d}x(V_2), \ldots, \mathrm{d}/\mathrm{d}x(V_n)$ are determined from n linear algebraic equations. These equations can be written in matrix form, namely:

$$\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \dots & \frac{dy_n}{dx} \\ \dots & \dots & \dots & \dots \\ \frac{d^{n-1}y_1}{dx} & \frac{d^{n-1}y_2}{dx} & \dots & \frac{d^{n-1}y_n}{dx} \end{bmatrix} \begin{bmatrix} \frac{dV_1}{dx} \\ \frac{dV_2}{dx} \\ . \\ \frac{dV_n}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ f(x) \end{bmatrix} , \tag{94}$$

or

$$\left[A\right] \left[\frac{dV}{dx}\right] = \left[G\right] .$$

Then matrix operations can be used to solve for dV_n/dx:

$$\left[\frac{dV}{dx}\right] = \left[A^{-1}\right] \left[G\right] . \tag{95}$$

The matrix V can be calculated from the following equations:

$$\left[V\right] = \int_{x_0}^{x} \left[A^{-1}\right] \left[G\right] dx . \tag{96}$$

To use the above method, the solution to eqs. (62) and (83) has the form

$$\widetilde{Z} = \widetilde{C}_1\widetilde{Z}_1 + \widetilde{C}_2\widetilde{Z}_2 , \tag{97}$$

where

$$\widetilde{Z}_1 = ZR_1 + iZI_1 , \qquad \widetilde{Z}_2 = ZR_2 + iZI_2 . \tag{98}$$

Then the solution was substituted into eq. (94) which gave

$$\left[\widetilde{A}\right] \left[\frac{d\widetilde{F}}{d\phi}\right] = \left[G\right] , \tag{99}$$

where

$$\left[\widetilde{A}\right] = \begin{bmatrix} \widetilde{Z}_1 & \widetilde{Z}_2 \\ \frac{d\widetilde{Z}_1}{d\phi} & \frac{d\widetilde{Z}_2}{d\phi} \end{bmatrix} , \qquad \left[\frac{d\widetilde{F}}{d\phi}\right] = \begin{bmatrix} \frac{d\widetilde{V}_1}{d\phi} \\ \frac{d\widetilde{V}_2}{d\phi} \end{bmatrix} , \qquad \left[G\right] = \begin{bmatrix} 0 \\ f(x) \end{bmatrix} . \tag{100--102}$$

To solve the above matrix equation (99), it was transformed into a real equation from a complex one. This was done by the following method [28]. A complex matrix equation was represented by

$$\tilde{A}\,\tilde{X} = \tilde{F}\,, \tag{103}$$

where

$$\tilde{A} = M + \mathrm{i}N\,, \qquad \tilde{F} = S + \mathrm{i}T\,, \qquad \tilde{X} = Y + \mathrm{i}Z\,.$$

This led to two real equations:

$$MY - NZ = S\,, \qquad NY + MZ = T\,, \tag{104}$$

which when written in matrix form became

$$\begin{bmatrix} M & -N \\ N & M \end{bmatrix} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} S \\ T \end{bmatrix}. \tag{105}$$

Using the above transformation and eq. (97) and (98), eq. (99) can be rewritten as:

$$\begin{bmatrix} ZR_1 & ZR_2 & -ZI_1 & -ZI_2 \\ \frac{\mathrm{d}}{\mathrm{d}\phi}(ZR_1) & \frac{\mathrm{d}}{\mathrm{d}\phi}(ZR_2) & -\frac{\mathrm{d}}{\mathrm{d}\phi}(ZI_1) & -\frac{\mathrm{d}}{\mathrm{d}\phi}(ZI_2) \\ ZI_1 & ZI_2 & ZR_1 & ZR_2 \\ \frac{\mathrm{d}}{\mathrm{d}\phi}(ZI_1) & \frac{\mathrm{d}}{\mathrm{d}\phi}(ZI_2) & \frac{\mathrm{d}}{\mathrm{d}\phi}(ZR_1) & \frac{\mathrm{d}}{\mathrm{d}\phi}(ZR_2) \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}\phi}(VR_1) \\ \frac{\mathrm{d}}{\mathrm{d}\phi}(VR_2) \\ \frac{\mathrm{d}}{\mathrm{d}\phi}(VI_1) \\ \frac{\mathrm{d}}{\mathrm{d}\phi}(VI_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{bmatrix}, \tag{106}$$

where

$$\frac{\mathrm{d}\tilde{V}_1}{\mathrm{d}\phi} = \frac{\mathrm{d}(VR_1)}{\mathrm{d}\phi} + \mathrm{i}\,\frac{\mathrm{d}(VI_1)}{\mathrm{d}\phi}\,, \qquad \frac{\mathrm{d}\tilde{V}_2}{\mathrm{d}\phi} = \frac{\mathrm{d}(VR_2)}{\mathrm{d}\phi} + \mathrm{i}\,\frac{\mathrm{d}(VI_2)}{\mathrm{d}\phi}\,. \tag{107}$$

After solving eq. (106) for the $\mathrm{d}V/\mathrm{d}\phi$ terms (which was done on the computer), the functions V were found by

$$\tilde{V}_1 = \int_{\phi_0}^{\phi} \frac{\mathrm{d}(VR_1)}{\mathrm{d}\phi}\,\mathrm{d}\phi + \mathrm{i}\int_{\phi_0}^{\phi} \frac{\mathrm{d}(VI_1)}{\mathrm{d}\phi}\,\mathrm{d}\phi\,, \qquad \tilde{V}_2 = \int_{\phi_0}^{\phi} \frac{\mathrm{d}(VR_2)}{\mathrm{d}\phi}\,\mathrm{d}\phi + \mathrm{i}\int_{\phi_0}^{\phi} \frac{\mathrm{d}(VI_2)}{\mathrm{d}\phi}\,\mathrm{d}\phi\,. \tag{108}$$

The particular solutions to the eqs. (62) and (83) were then written as

$$\tilde{Z}_{\mathrm{P}} = \tilde{V}_1\tilde{Z}_1 + \tilde{V}_2\tilde{Z}_2\,. \tag{109}$$

The stress resultants for the particular solution were derived in the same manner and form as in the bending theories.

4. Discontinuity relations between the shell segments

The configuration analyzed is shown in fig. 6, where the positive directions for the stress resultants, slopes, and deflections are shown. In this analysis, continuity at the intersections of the midsurfaces of the shell element is maintained.

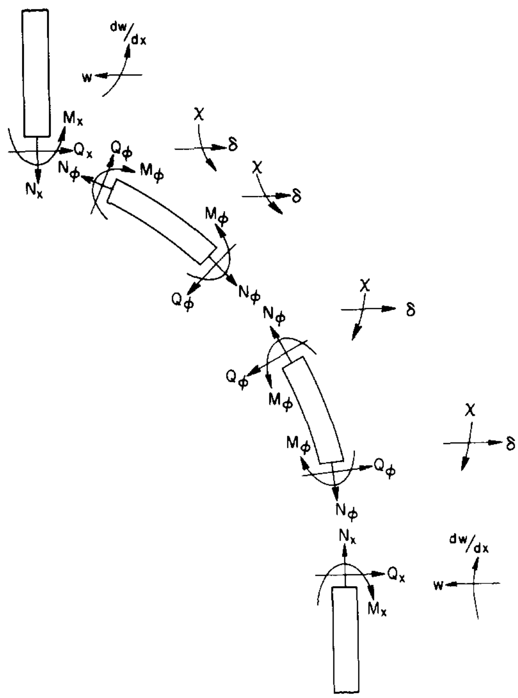

Fig. 6. Discontinuity diagram of the shell elements.

4.1. *The axisymmetric case*

The deflections at the junctions of the nozzle and sphere are equal

$$w_n + \delta_s = 0 . \tag{110}$$

Summing the moments at this junction yields

$$M_{xn} - M_{\phi s} = 0 . \tag{111}$$

A summation of the forces in the direction of $Q_{\phi s}$ gives

$$Q_{xn} \sin \phi_o - N_{xn} \cos \phi_o + Q_{\phi s} = 0 . \tag{112}$$

The angles of rotation of meridional tangents are equal; thus

$$\frac{dw_n}{dx} - X_s = 0 \,. \tag{113}$$

At the junction between the sphere and torus, the following equations are derived:

$$Q_{\phi s} + Q_{\phi t} = 0 \,, \qquad M_{\phi s} + M_{\phi t} = 0 \,, \qquad \delta_s - \delta_t = 0 \,, \qquad X_s + X_t = 0 \,. \tag{114}$$

The equations derived at the junction of the torus and cylinder are as follows:

$$\delta_t + W_c = 0 \,, \qquad X_t - \frac{dW_c}{dx} = 0 \,, \qquad M_{\phi t} + M_{xc} = 0 \,, \qquad Q_{\phi t} + Q_{xc} = 0 \,. \tag{115}$$

The above equations [eqs. (110) through (115)] were solved simultaneously and gave the unknown coefficients needed in the bending solutions to calculate the stresses and stress resultants.

4.2. *The nonaxisymmetric case*

For complete solutions to shell problems, we must include body movements as well as edge bending and membrane displacements [29, 30]. For this case, first consider the spherical shell having displacements due to edge bending and membrane solutions: U_{1s}, V_{1s}, W_{1s}, and X_{1s} as shown in fig. 7.

In cylindrical coordinates, the displacements of the edge of the shell are completely defined by the following: U_{1s}, δ_{1hs}, δ_{1vs}, and V_{1s}, where

$$\delta_{1hs} = W_{1s} \sin\phi_o + V_{1s} \cos\phi_o \,, \qquad \delta_{1vs} = -W_{1s} \cos\phi_o + V_{1s} \sin\phi_o \,. \tag{116,117}$$

The total displacements consist of the displacements discussed above and the rigid body movements. The rigid body movements of the shell are (1) the shell moving parallel to the plane of its edges by an amount h_{1s} and (2) rotating about this point through the angle V_{1s}/b as shown in fig. 7. Therefore, the total displacements are:

$$U_s = U_{1s} - h_{1s} \,, \qquad \delta_{hs} = \delta_{1hs} + h_{1s} \,, \qquad \delta_{vs} = \delta_{1vs} + V_{1s} \,, \qquad X_s = X_{1s} - \frac{V_{1s}}{b} \,. \tag{118}$$

In the same way the displacements for the nozzle are derived as follows:

$$U_n = U_{1n} - h_{1n} \,, \qquad \delta_{hn} = \delta_{1hn} + h_{1n} \,, \qquad \delta_{vn} = \delta_{1vn} + V_{1n} \,, \qquad X_n = X_{1n} - \frac{V_{1s}}{b} \,. \tag{119}$$

Compatibility between the nozzle and spherical shell requires that

$$U_n = U_s \,, \qquad \delta_{hn} = \delta_{hs} \,, \qquad \delta_{vn} = \delta_{vs} \,, \qquad X_n = X_s \,. \tag{120}$$

The first two compatibility conditions are combined by eliminating the h_1 terms:

$$\delta_{1hs} + U_{1s} = \delta_{1hn} + U_{1n} \,. \tag{121}$$

Also, by eliminating the V_1 terms, the remaining compatibility equations yield:

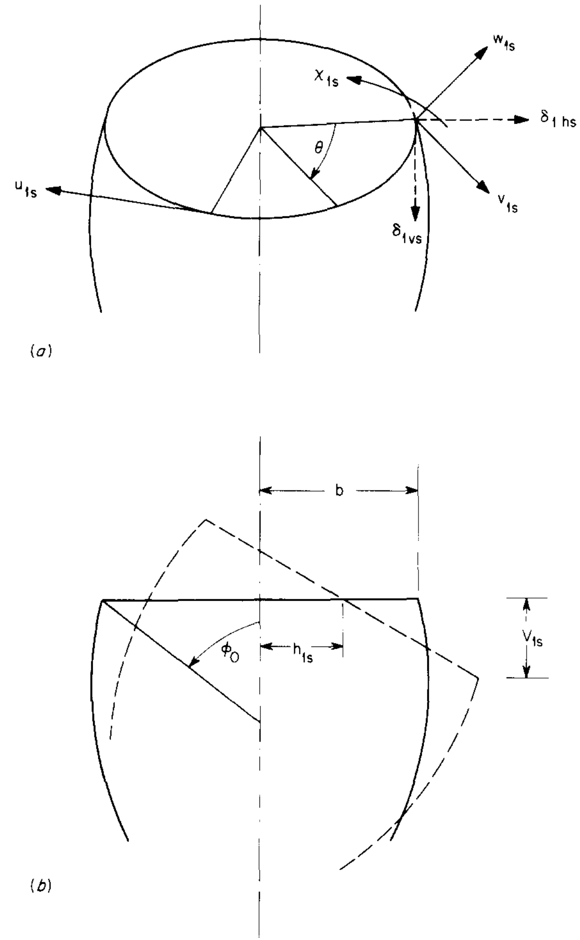

Fig. 7. Spherical shell displacements.

$$X_{1s} + \frac{\delta_{1vs}}{b} = X_{1n} + \frac{\delta_{1vn}}{b} . \tag{122}$$

Therefore, the rigid body movements have been elimated and the compatibility conditions are satisfied by the two equations which involve only the bending and the membrane solutions. The above analyses are derived in refs. [29] and [30].

In a similar manner, equations can be derived which satisfy compatibility between the spherical and toroidal shells and between the toroidal and cylindrial shells. The δ_v's for the nozzle and cylinder are very small and, therefore, were neglected.

The discontinuity relations between the shell segments for this loading case are:

$$W_n + \delta_s = 0\,, \qquad M_{xn} - M_{\phi s} = 0\,, \qquad Q_{xn} + Q_{\phi s} \sin\phi - N_{\phi s}\cos\phi = 0\,,$$

$$\frac{dw_n}{dx} - X_s + \frac{\delta_{vs}}{R_s \sin\phi}\,, \qquad Q_{\phi s} + Q_{\phi t} = 0\,, \qquad X_s + \frac{\delta_{vs}}{R_s \sin\phi} + X_t + \frac{\delta_{vt}}{R_o(1+\alpha\sin\phi)} = 0\,,$$

$$M_{\phi s} + M_{\phi t} = 0\,, \qquad \delta_s - \delta_t = 0\,, \qquad \delta_t + W_c = 0\,,$$

$$X_t + \frac{\delta_{vt}}{R_o(1+\alpha\sin\phi)} - \frac{dW_c}{dx} = 0\,, \qquad M_{\phi t} + M_{xc} = 0\,, \qquad Q_{\phi t} + Q_{xc} = 0\,. \tag{123}$$

In these equations, the shear terms $Q_{\phi s}$ and $Q_{\phi t}$ are the Kirchoff shear forces for the sphere and torus.

5. Discussion and comparison of results

A computer program was written to simultaneously solve the equations derived in section 4 for the unknown constants. The unknown constants were then used to calculate stresses in the structures, and analytical predictions were compared with the results from two experimental shell model tests to illustrate the accuracy of the analyses. The complete equations used for all the shell structures are given in ref. [31] along with a flow diagram of the computer program. A sketch of the first model is shown in fig. 8. This is a steel model and is identified as no. 32 in ref. [32]. In this case, the experimental data were normalized by dividing all the stresses by the circumferential membrane stress of the cylindrical shell. The theoretical analysis was done by assuming a very small membrane nozzle

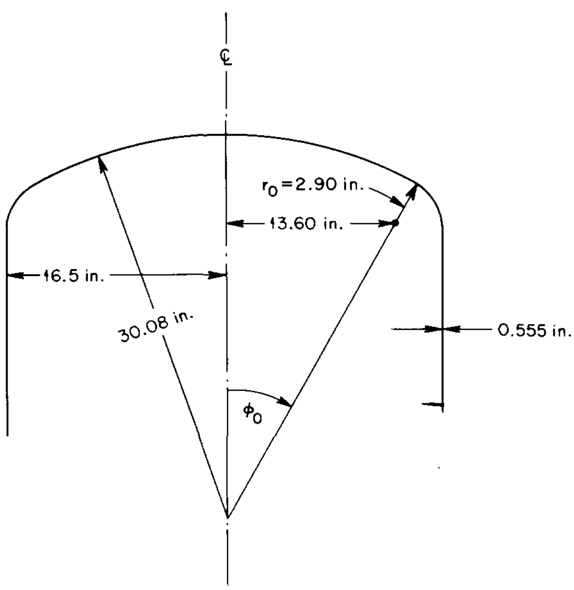

Fig. 8. Sketch of the Findly, Moffat, and Stanley model no. 32.

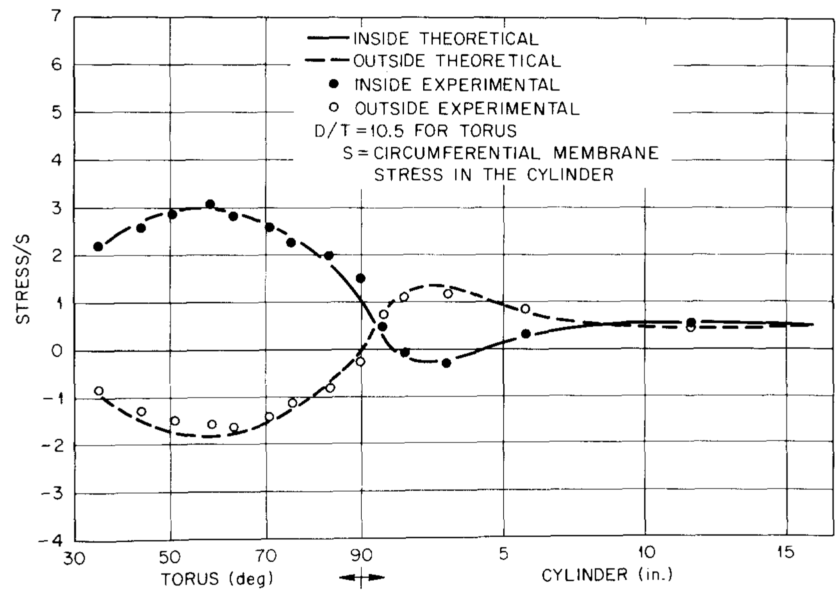

Fig. 9. Findlay, Moffat, and Stanley model no. 32; pressure loading, meridional stress.

attached to the torispherical shell at its apex, since the computer program was written to analyze a radial nozzle attached to a torispherical shell, which in turn was attached to a cylindrical shell.

Comparisons of theoretical and experimental results are shown in figs. 9 and 10. Results for the toroidal and cylindrical shells only are shown since the membrane nozzle attached to the spherical portion would give a different stress distribution in the sphere close to the nozzle. The meridional stresses are shown in fig. 9, and excellent agreement is shown between the results. The circumferential stresses are shown in fig. 10 and very good comparison is again found.

The second model chosen for analysis is sketched in fig. 11. This model was also steel and the experimental results are given in refs. [33] and [34]. This model was analyzed in the same manner as the first model.

Comparisons of experimental and theoretical results are shown in figs. 12 and 13. Again, results for only the toroidal and cylindrical shells are shown. The meridional stresses are shown in fig. 12 and excellent agreement is obtained between the results. The circumferential stresses are shown in fig. 13 and very good comparison is again shown.

The comparison just discussed illustrate the accuracy of the analysis for the axisymmetric internal pressure case. No experimental data for concentrated loads was found in the literature and therefore no direct comparisons can be made for these loadings. However, since the analysis and equations for the concentrated axial load applied to the nozzle are similar to those for pressure, it is not unreasonable to assume that the analysis for the concentrated axial load applied to the nozzle is correct.

To study the effect of an external moment applied to a nozzle attached to torispherical shell, comparisons between axial thrust and bending loads were made. The configuration which was analyzed had the following parameters:

(a) Radius of nozzle = 4.17 in.
(b) Thickness of nozzle = 0.83 in.
(c) Radius of sphere = 24 in.
(d) Thickness of sphere = 0.50 in.
(e) Radius of torus = 2.0 in.

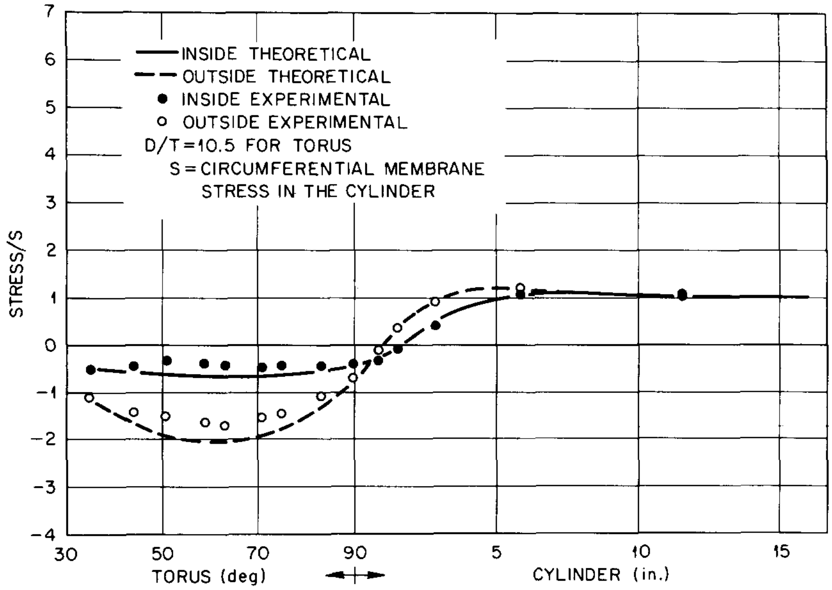

Fig. 10. Findlay, Moffat, and Stanley model no. 32; pressure loading, circumferential stress.

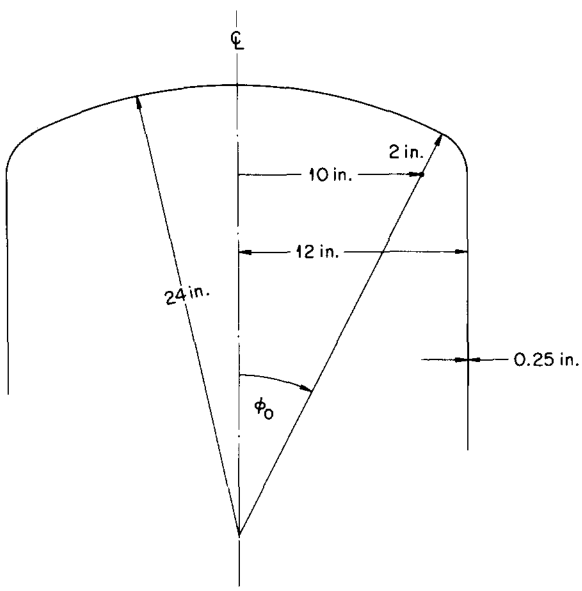

Fig. 11. Sketch of the Stoddart and Owen model.

(f) Thickness of torus = 0.50 in.
(g) Radius of cylinder = 12.0 in.
(h) Thickness of cylinder = 0.5 in.
Comparisons between the two loads for the torispherical shell are shown in figs. 14 through 17. In these analyses,

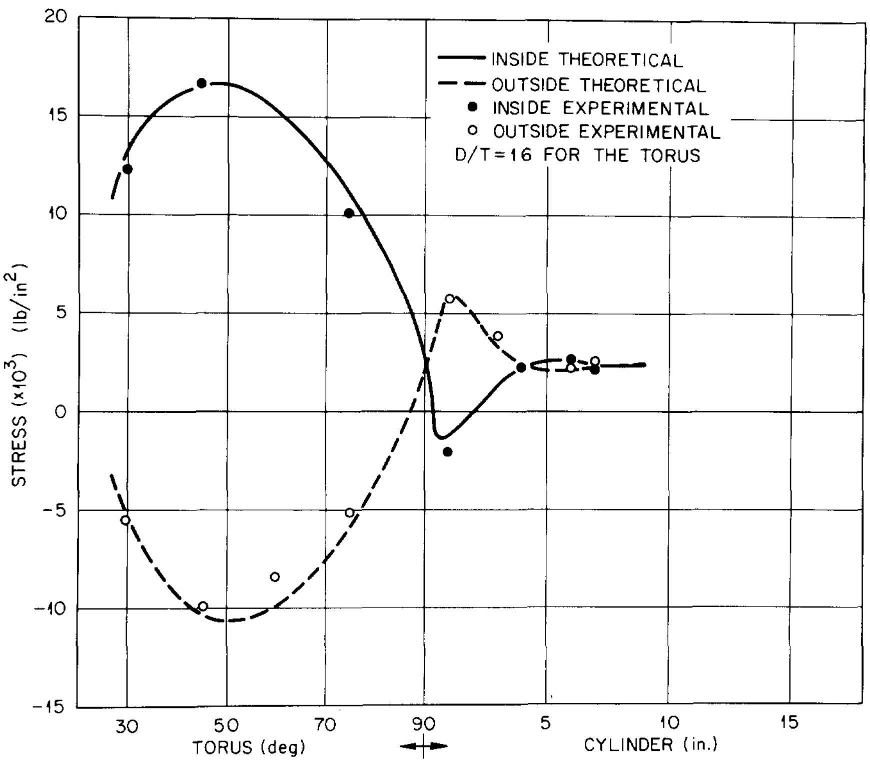

Fig. 12. Stoddart and Owen model; pressure loading, meridional stress.

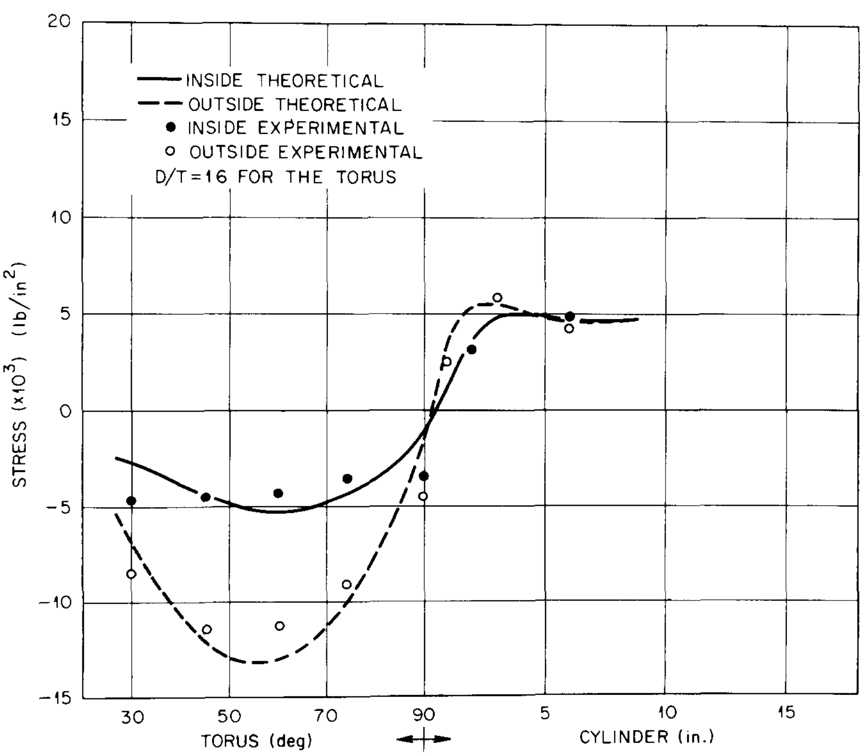

Fig. 13. Stoddart and Owen model; pressure loading, circumferential stress.

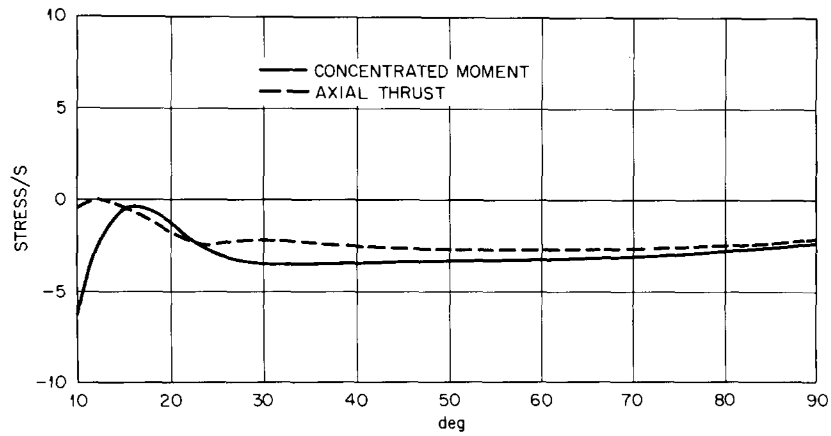

Fig. 14. Comparison between axial thrust and concentrated moment for test model; circumferential stress, inside surface.

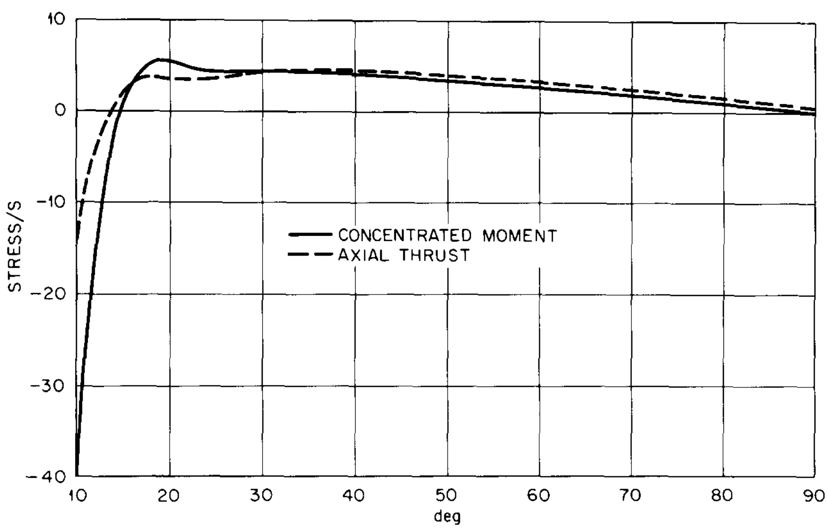

Fig. 15. Meridional stress, inside surface.

the membrane stresses in the cylindrical shell were held the same for both loadings in the plane of the moment, $\theta = 0$, so that the comparison could be made between the stress states. As can be seen, the stresses in the toroidal part of the shell were about the same for both loadings, while great differences in the stress levels can be seen in the spherical part of the shell. The reasons for the close comparison of the stresses in the toroidal part are that the form of the stress resultants in the concentrated moment loading are different from the axial load case by a term multiplied by $(t/r_0)^2/12(1-\nu^2)$. Since this term is very small and the stress state in the cylindrical shell at $\theta = 0$ is the same for both loading cases, the stress should be approximately the same. When comparisons between the states of stress in a torispherical shell caused by concentrated loads and internal pressure are made, it is found that under internal pressure loading the toroidal part of the shell contains the maximum stresses while for concentrated loads the maximum stress occurs in the spherical portion of the shell.

The buckling of torispherical shells under internal pressure is a phenomenon that designers have been warned of in the past. The model which was used in ref. 2 for a buckling experiment was analyzed. A maximum stress of

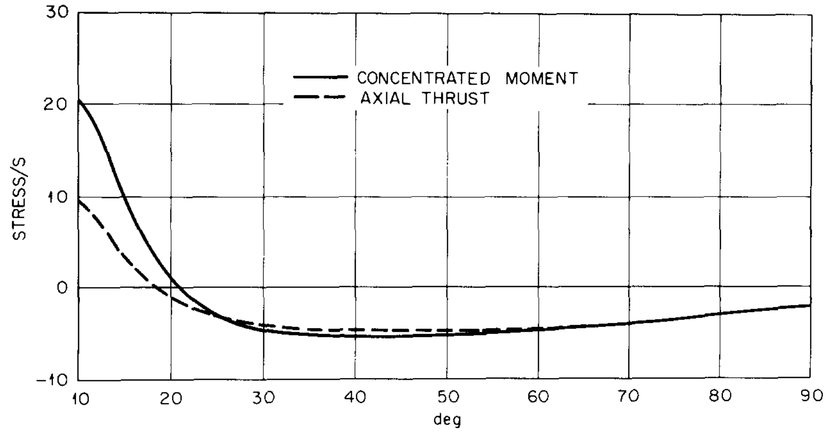

Fig. 16. Circumferential stress, outside surface.

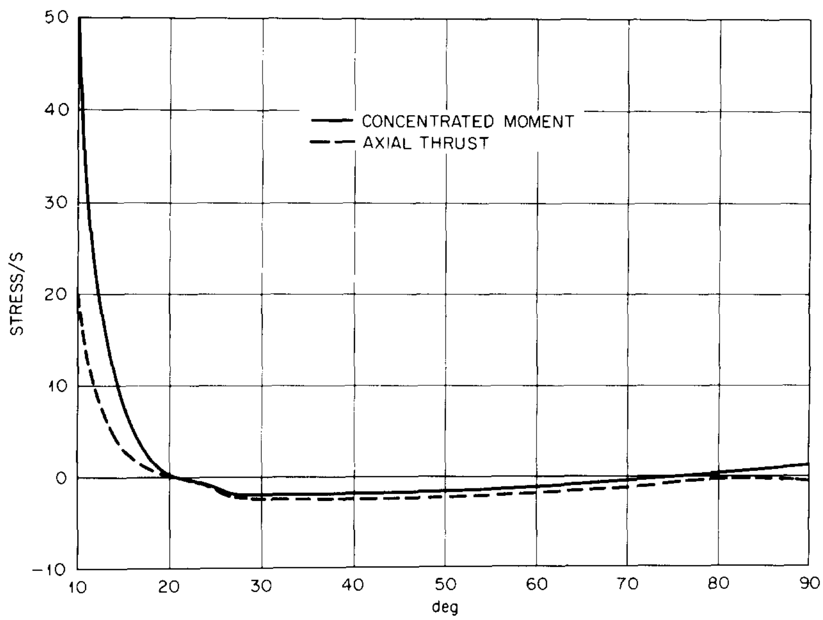

Fig. 17. Meridional stress, outside surface.

32 580 psi tension was calculated on the inside surface of the torus in the meridional direction. The yield stress for this model was placed at 35 000 psi, which could mean that the inside surface of the model yielded. This yielding coupled with high compressive circumferential stresses on the outside surface could cause the torus to "buckle". Dimples were formed in the torus with the fold lines in the meridional direction and the outside surface of the torus was completely dimpled all around its circumference. Therefore, the buckling of the torus could actually be a plastic collapse of the structure; and to avoid "buckling" of this type structure for any loading, the stress levels must be kept well below the yield stress.

6. Conclusions

Methods were developed to analyze torispherical shells loaded under internal pressure, an axial thrust to a radial nozzle attached to the apex of the shell, and a moment applied to the radial nozzle. A particular solution to the bending equation was developed for both the internal pressure and external loading cases. Solutions to the bending equation for the external cases were developed. Membrane solutions were developed for all these loading cases.

The analysis for internal pressure was compared with experimental results and was found to be very accurate. Since the analyses for the other loads are very similar to that for internal pressure loading, it was reasonable to assume that they also are accurate. However, the accuracy of these loadings could not be proved since no experimental data for such loadings were found in the literature.

The "buckling" of torispherical shells under internal pressure seemed actually to be a plastic collapse of the structure. Therefore, no "buckling" of the torispherical shell under external loads would be expected if the shells are kept within the elastic stress limits.

Notation

a	= radius of the nozzle or cylindrical shell,
C	= constant used in analysis of toroidal shell segment similar to flexural rigidity, $C = t/\sqrt{12(1-\nu^2)}$,
$\widetilde{C}_1, \widetilde{C}_2$	= complex constants used in the calculation of stresses and stress resultants in the toroidal shell segment,
D	= extensional rigidity of spherical shell segment, $D = Et/(1-\nu^2)$,
E	= modulus of elasticity [psi].
F	= axial force applied to a shell segment or nozzle [lb],
$H_{\frac{1}{3}}^{(1)}, H_{\frac{1}{3}}^{(2)}$	= Hankel functions of $\frac{1}{3}$ order, first and second kind,
$\mathrm{In}^{3}(\widetilde{Z})^{3}$	= imaginary part of the complex solution function for the toroidal shell,
k_ϕ, k_θ	= change in curvature in the meridional and circumferential directions of a toroidal shell,
K	= flexural rigidity of shells, $K = Et/12(1-\nu^2)$,
m	= Fourier harmonic number,
M	= external or concentrated moment applied to the nozzle,
$M_\phi, M_\theta, M_{\phi\theta}$	= meridional, circumferential, and twisting moments in spherical and toroidal shell segments [lb–in/in],
M_X, M_θ	= axial and circumferential moments in the nozzle and cylinder [lb–in/in],
$\widetilde{N}$	= combination complex stress resultants of $\widetilde{N}_\phi + \widetilde{N}_\theta$,
$N_\phi, N_\theta, N_{\phi\theta}$	= meridional, circumferential, and twisting membrane stress resultants in spherical and toroidal shell segments [lb/in],
$\widetilde{N}_\phi, \widetilde{N}_\theta, \widetilde{N}_{\phi\theta}$	= meridional, circumferential, and twisting complex stress resultants in a toroidal shell segment,
N_x, N_θ	= axial and circumferential membrane stress resultants in the nozzle and cylinder [lb/in],
N_ϕ^*, N_θ^*	= known membrane stress resultants in spherical and toroidal shell segments,
P	= internal pressure [psi],
P_n, P_ϕ, P_θ	= surface loading components normal to the surface, in the meridional direction, and in the circumferential direction,
Q_ϕ, Q_θ	= meridional and circumferential shear stress resultants in toroidal and spherical shell segments,
Q_X	= axial shear stress resultant in the nozzle and cylindrical shell,
r_o	= radius of the circular cross section of the toroidal shell,
R_o	= the distance of the center of the cross section from the axis of revolution of the toroidal shell,
R_s	= radius of sphere,

R_1	= principal radius of curvature of a shell in the meridional direction,
R_2	= second principal radius of curvature of a shell in circumferential direction,
$\mathrm{Re}\,(\widetilde{Z})$	= real part of the complex solution function for the toroidal shell,
t	= shell thickness [in],
w	= normal deflection of the nozzle of cylindrical shell [in],
$\widetilde{Z}$	= bending solution function for a toroidal shell segment,
$\widetilde{Z}_p$	= particular bending solution for a toroidal shell segment,
$dw/d\phi$	= slope or rotation of the tangent to the axial direction of a nozzle or cylindrical shell,
$2d^2$	= constants used in analysis of toroidal shell = $\sqrt{12(1-\nu^2)}\, r_0^2/R_0 t$,
α	= constant used in analysis of toroidal shell = r_0/R_0,
δ	= horizontal deflection of a spherical or toroidal shell,
δ_v	= vertical deflection of a spherical or toroidal shell,
θ	= circumferential angle of shells,
ν	= Poisson's ratio,
ϕ	= meridional angle of shells,
ϕ_0	= top meridional angle of the torus,
X_ϕ, X_θ	= slope of shells in the meridional and circumferential directions,
$(\tilde{\ })$	= complex variable,
$(\ )_N$	= nozzle stress resultants and deflections,
$(\ )_S, (\ )^S$	= spherical shell stress resultants and deflections,
$(\ )_t$	= toroidal shell stress resultants and deflections,
$(\ )_C$	= cylindrical shell stress resultants and deflections,

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