TUTORIAL No. 1

FLUID FLOW THEORY

In order to complete this tutorial you should already have completed level 1 or have a good basic knowledge of fluid mechanics equivalent to the Engineering Council part 1 examination 103.

When you have completed this tutorial, you should be able to do the following.

- Explain the meaning of viscosity.
- Define the units of viscosity.
- Describe the basic principles of viscometers.
- Describe non-Newtonian flow
- Explain and solve problems involving laminar flow though pipes and between parallel surfaces.
- Explain and solve problems involving drag force on spheres.
- Explain and solve problems involving turbulent flow.
- Explain and solve problems involving friction coefficient.

Throughout there are worked examples, assignments and typical exam questions. You should complete each assignment in order so that you progress from one level of knowledge to another.

Let us start by examining the meaning of viscosity and how it is measured.

1. VISCOSITY

1.1 BASIC THEORY

Molecules of fluids exert forces of attraction on each other. In liquids this is strong enough to keep the mass together but not strong enough to keep it rigid. In gases these forces are very weak and cannot hold the mass together.

When a fluid flows over a surface, the layer next to the surface may become attached to it (it wets the surface). The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid.

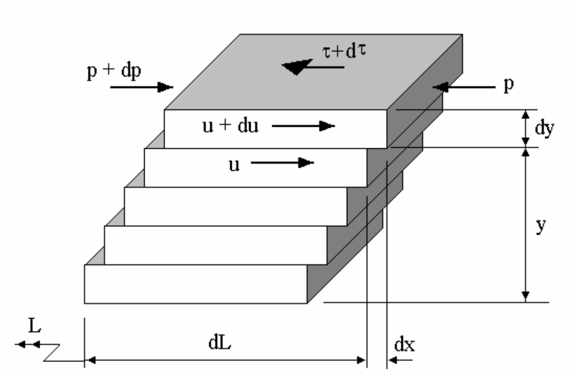


Fig.1.1

Let us suppose that the fluid is flowing over a flat surface in laminated layers from left to right as shown in figure 1.1.

y is the distance above the solid surface (no slip surface)
L is an arbitrary distance from a point upstream.
dy is the thickness of each layer.
dL is the length of the layer.
dx is the distance moved by each layer relative to the one below in a corresponding time dt.
u is the velocity of any layer.
du is the increase in velocity between two adjacent layers.

Each layer moves a distance dx in time dt relative to the layer below it. The ratio dx/dt must be the change in velocity between layers so du = dx/dt.

When any material is deformed sideways by a (shear) force acting in the same direction, a shear stress τ is produced between the layers and a corresponding shear strain γ is produced. Shear strain is defined as follows.

$$\gamma = \frac{\text{sideways deformation}}{\text{height of the layer being deformed}} = \frac{dx}{dy}$$

The rate of shear strain is defined as follows.

$$\dot{\gamma} = \frac{\text{shear strain}}{\text{time taken}} = \frac{\gamma}{dt} = \frac{dx}{dt\,dy} = \frac{du}{dy}$$

It is found that fluids such as water, oil and air, behave in such a manner that the shear stress between layers is directly proportional to the rate of shear strain.

$$\tau = \text{constant x } \dot{\gamma}$$

Fluids that obey this law are called ***NEWTONIAN FLUIDS***.

It is the constant in this formula that we know as the dynamic viscosity of the fluid.

DYNAMIC VISCOSITY $\boldsymbol{\mu}$ **=** $\dfrac{\text{shear stress}}{\text{rate of shear}} = \dfrac{\tau}{\dot{\gamma}} = \tau\dfrac{dy}{du}$

FORCE BALANCE AND VELOCITY DISTRIBUTION

A shear stress τ exists between each layer and this increases by dτ over each layer. The pressure difference between the downstream end and the upstream end is dp.

The pressure change is needed to overcome the shear stress. The total force on a layer must be zero so balancing forces on one layer (assumed 1 m wide) we get the following.

$$dp\,dy + d\tau\,dL = 0$$

$$\frac{d\tau}{dy} = -\frac{dp}{dL}$$

It is normally assumed that the pressure declines uniformly with distance downstream so the ***pressure gradient*** $\frac{dp}{dL}$ is assumed constant. The minus sign indicates that the pressure falls with distance. Integrating between the no slip surface (y = 0) and any height y we get

$$-\frac{dp}{dL} = \frac{d\tau}{dy} = \frac{d\left(\mu\frac{du}{dy}\right)}{dy}$$

$$-\frac{dp}{dL} = \mu\frac{d^2u}{dy^2}..............(1.1)$$

Integrating twice to solve u we get the following.

$$-y\frac{dp}{dL} = \mu\frac{du}{dy} + A$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + Ay + B$$

A and B are constants of integration that should be solved based on the known conditions (boundary conditions). For the flat surface considered in figure 1.1 one boundary condition is that u = 0 when y = 0 (the no slip surface). Substitution reveals the following.

0 = 0 +0 +B   hence B = 0

At some height δ above the surface, the velocity will reach the mainstream velocity u_o. This gives us the second boundary condition $u = u_o$ when y = δ.

Substituting we find the following.

$$-\frac{\delta^2}{2}\frac{dp}{dL} = \mu u_o + A\delta$$

$$A = -\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta} \quad \text{hence}$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + \left(-\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta}\right)y$$

$$u = y\left(\frac{\delta}{2\mu}\frac{dp}{dL} + \frac{u_o}{\delta}\right)$$

Plotting u against y gives figure 1.2.

BOUNDARY LAYER.

The velocity grows from zero at the surface to a maximum at height δ. In theory, the value of δ is infinity but in practice it is taken as the height needed to obtain 99% of the mainstream velocity. This layer is called the boundary layer and δ is the boundary layer thickness. It is a very important concept and is discussed more fully in later work. The inverse gradient of the boundary layer is du/dy and this is the rate of shear strain γ.

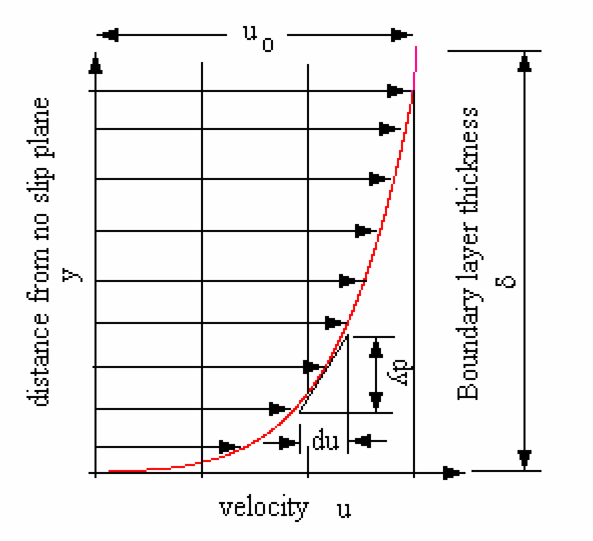


Fig.1.2

1.2. UNITS of VISCOSITY

1.2.1 DYNAMIC VISCOSITY μ

The units of dynamic viscosity μ are N s/m^2. It is normal in the international system (SI) to give a name to a compound unit. The old metric unit was a dyne.s/cm^2 and this was called a POISE after Poiseuille. The SI unit is related to the Poise as follows.

10 Poise = 1 Ns/m^2 which is not an acceptable multiple. Since, however, 1 Centi Poise (1cP) is 0.001 N s/m^2 then the cP is the accepted SI unit.

1cP = 0.001 N s/m^2.

The symbol η is also commonly used for dynamic viscosity.

There are other ways of expressing viscosity and this is covered next.

1.2.2 KINEMATIC VISCOSITY ν

This is defined as : ν = dynamic viscosity /density

$$\nu = \mu/\rho$$

The basic units are m^2/s. The old metric unit was the cm^2/s and this was called the STOKE after the British scientist. The SI unit is related to the Stoke as follows.

1 Stoke (St) = 0.0001 m^2/s and is not an acceptable SI multiple. The centi Stoke (cSt),however, is 0.000001 m^2/s and this is an acceptable multiple.

1cSt = 0.000001 m^2/s = 1 mm^2/s

1.2.3 OTHER UNITS

Other units of viscosity have come about because of the way viscosity is measured. For example REDWOOD SECONDS comes from the name of the Redwood viscometer. Other units are Engler Degrees, SAE numbers and so on. Conversion charts and formulae are available to convert them into useable engineering or SI units.

1.2.4 VISCOMETERS

The measurement of viscosity is a large and complicated subject. The principles rely on the resistance to flow or the resistance to motion through a fluid. Many of these are covered in British Standards 188. The following is a brief description of some types.

U TUBE VISCOMETER

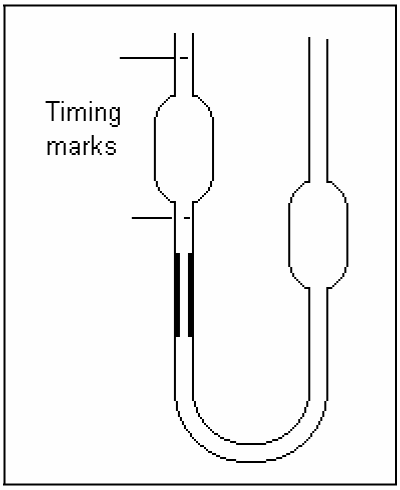


Fig.1.3

The fluid is drawn up into a reservoir and allowed to run through a capillary tube to another reservoir in the other limb of the U tube.

The time taken for the level to fall between the marks is converted into cSt by multiplying the time by the viscometer constant.

$$\nu = ct$$

The constant c should be accurately obtained by calibrating the viscometer against a master viscometer from a standards laboratory.

REDWOOD VISCOMETER

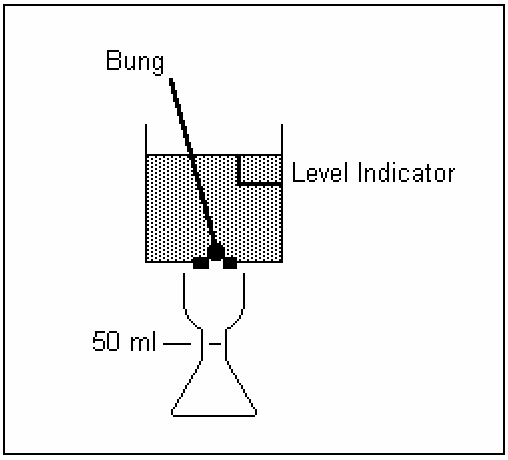
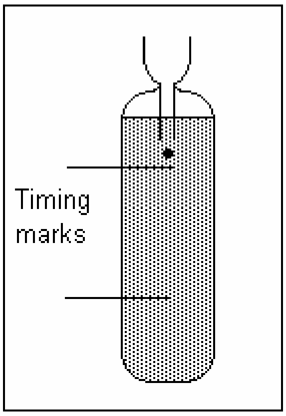


Fig.1.4

This works on the principle of allowing the fluid to run through an orifice of very accurate size in an agate block.

50 ml of fluid are allowed to fall from the level indicator into a measuring flask. The time taken is the viscosity in Redwood seconds. There are two sizes giving Redwood No.1 or No.2 seconds. These units are converted into engineering units with tables.

FALLING SPHERE VISCOMETER



This viscometer is covered in BS188 and is based on measuring the time for a small sphere to fall in a viscous fluid from one level to another. The buoyant weight of the sphere is balanced by the fluid resistance and the sphere falls with a constant velocity. The theory is based on Stokes' Law and is only valid for very slow velocities. The theory is covered later in the section on laminar flow where it is shown that the terminal velocity (u) of the sphere is related to the dynamic viscosity (μ) and the density of the fluid and sphere (ρ_f and ρ_s) by the formula

$$\mu = F \; gd^2(\rho_s - \rho_f)/18u$$

Fig.1.5

F is a correction factor called the Faxen correction factor, which takes into account a reduction in the velocity due to the effect of the fluid being constrained to flow between the wall of the tube and the sphere.

ROTATIONAL TYPES

There are many types of viscometers, which use the principle that it requires a torque to rotate or oscillate a disc or cylinder in a fluid. The torque is related to the viscosity. Modern instruments consist of a small electric motor, which spins a disc or cylinder in the fluid. The torsion of the connecting shaft is measured and processed into a digital readout of the viscosity in engineering units.

You should now find out more details about viscometers by reading BS188, suitable textbooks or literature from oil companies.

ASSIGNMENT No. 1

1. Describe the principle of operation of the following types of viscometers.

a. Redwood Viscometers.

b. British Standard 188 glass U tube viscometer.

c. British Standard 188 Falling Sphere Viscometer.

d. Any form of Rotational Viscometer

Note that this covers the E.C. exam question 6a from the 1987 paper.

2. LAMINAR FLOW THEORY

The following work only applies to Newtonian fluids.

2.1 LAMINAR FLOW

A ***stream line*** is an imaginary line with no flow normal to it, only along it. When the flow is laminar, the streamlines are parallel and for flow between two parallel surfaces we may consider the flow as made up of parallel laminar layers. In a pipe these laminar layers are cylindrical and may be called ***stream tubes***. In laminar flow, no mixing occurs between adjacent layers and it occurs at low average velocities.

2.2 TURBULENT FLOW

The shearing process causes energy loss and heating of the fluid. This increases with mean velocity. When a certain critical velocity is exceeded, the streamlines break up and mixing of the fluid occurs. The diagram illustrates Reynolds coloured ribbon experiment. Coloured dye is injected into a horizontal flow. When the flow is laminar the dye passes along without mixing with the water. When the speed of the flow is increased turbulence sets in and the dye mixes with the surrounding water. One explanation of this transition is that it is necessary to change the pressure loss into other forms of energy such as angular kinetic energy as indicated by small eddies in the flow.

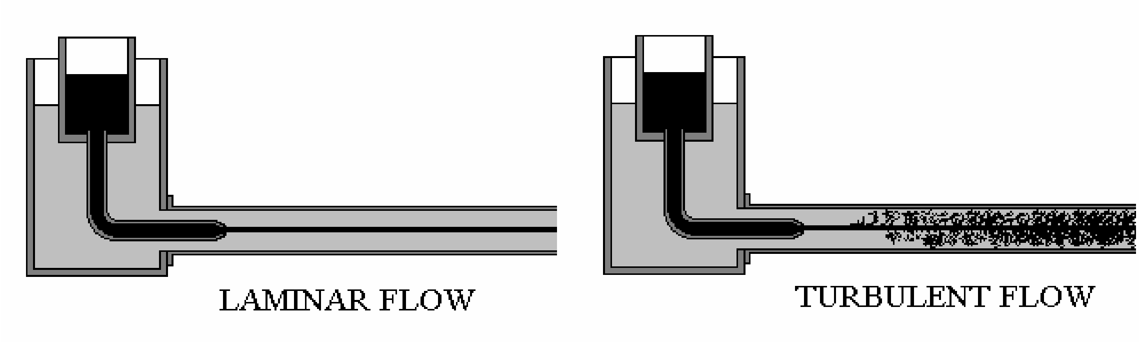


Fig.2.1

2.3 LAMINAR AND TURBULENT BOUNDARY LAYERS

In chapter 2 it was explained that a ***boundary layer*** is the layer in which the velocity grows from zero at the wall (no slip surface) to 99% of the maximum and the thickness of the layer is denoted δ. When the flow within the boundary layer becomes turbulent, the shape of the boundary layers waivers and when diagrams are drawn of turbulent boundary layers, the mean shape is usually shown. Comparing a laminar and turbulent boundary layer reveals that the turbulent layer is thinner than the laminar layer.

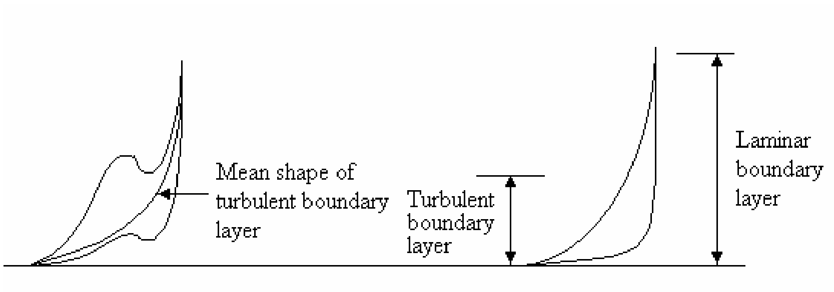


Fig.2.2

2.4 CRITICAL VELOCITY - REYNOLDS NUMBER

When a fluid flows in a pipe at a volumetric flow rate Q m^3/s the average velocity is defined $u_m = \frac{Q}{A}$ A is the cross sectional area.

The Reynolds number is defined as $R_e = \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu}$

If you check the units of R_e you will see that there are none and that it is a dimensionless number. You will learn more about such numbers in a later section.

Reynolds discovered that it was possible to predict the velocity or flow rate at which the transition from laminar to turbulent flow occurred for any Newtonian fluid in any pipe. He also discovered that the critical velocity at which it changed back again was different. He found that when the flow was gradually increased, the change from laminar to turbulent always occurred at a Reynolds number of 2500 and when the flow was gradually reduced it changed back again at a Reynolds number of 2000. Normally, 2000 is taken as the critical value.

WORKED EXAMPLE 2.1

Oil of density 860 kg/m^3 has a kinematic viscosity of 40 cSt. Calculate the critical velocity when it flows in a pipe 50 mm bore diameter.

SOLUTION

$$R_e = \frac{u_m D}{\nu}$$

$$u_m = \frac{R_e \nu}{D} = \frac{2000 x 40 x 10^{-6}}{0.05} = 1.6 \text{ m/s}$$

2.5 DERIVATION OF POISEUILLE'S EQUATION for LAMINAR FLOW

Poiseuille did the original derivation shown below which relates pressure loss in a pipe to the velocity and viscosity for LAMINAR FLOW. His equation is the basis for measurement of viscosity hence his name has been used for the unit of viscosity. Consider a pipe with laminar flow in it. Consider a stream tube of length ΔL at radius r and thickness dr.

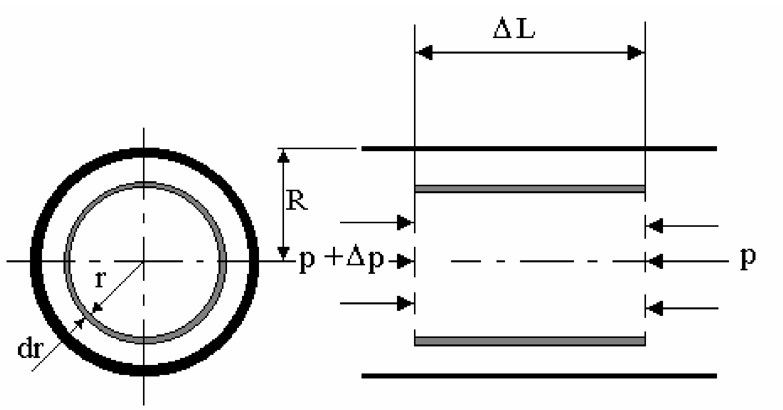


Fig.2.3

y is the distance from the pipe wall. $y = R - r$ $dy = -dr$ $\frac{du}{dy} = -\frac{du}{dr}$

The shear stress on the outside of the stream tube is τ. The force (F_s) acting from right to left is due to the shear stress and is found by multiplying τ by the surface area.

$F_s = \tau \times 2\pi r\,\Delta L$

For a Newtonian fluid , $\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$. Substituting for τ we get the following.

$$F_s = -2\pi r\Delta L\mu\frac{du}{dr}$$

The pressure difference between the left end and the right end of the section is Δp. The force due to this (F_p) is Δp x circular area of radius r.

$F_p = \Delta p \times \pi r^2$

Equating forces we have $-2\pi r\mu\Delta L\frac{du}{dr} = \Delta p\pi r^2$

$$du = -\frac{\Delta p}{2\mu\Delta L}rdr$$

In order to obtain the velocity of the streamline at any radius r we must integrate between the limits u = 0 when r = R and u = u when r = r.

$$\int_0^u du = -\frac{\Delta p}{2\mu\Delta L}\int_R^r rdr$$

$$u = -\frac{\Delta p}{4\mu\Delta L}\left(r^2 - R^2\right)$$

$$u = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right)$$

This is the equation of a Parabola so if the equation is plotted to show the boundary layer, it is seen to extend from zero at the edge to a maximum at the middle.

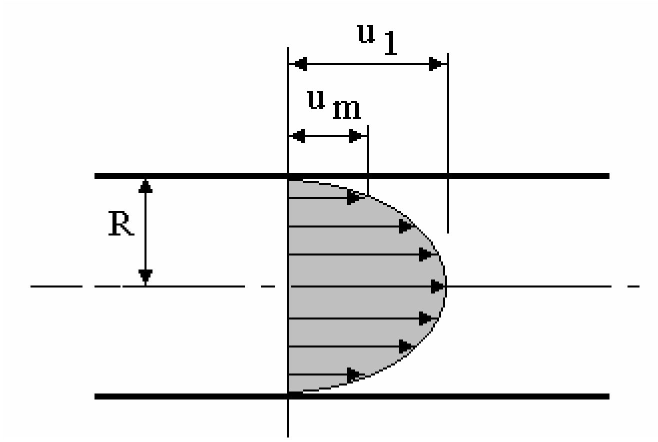


Fig.2.4

For maximum velocity put r = 0 and we get $u_1 = \frac{\Delta p R^2}{4\mu \Delta L}$

The average height of a parabola is half the maximum value so the average velocity is

$$u_m = \frac{\Delta p R^2}{8\mu \Delta L}$$

Often we wish to calculate the pressure drop in terms of diameter D. Substitute R=D/2 and rearrange.

$$\Delta p = \frac{32\mu \Delta L u_m}{D^2}$$

The volume flow rate is average velocity x cross sectional area.

$$Q = \frac{\pi R^2 \Delta p R^2}{8\mu \Delta L} = \frac{\pi R^4 \Delta p}{8\mu \Delta L} = \frac{\pi D^4 \Delta p}{128\mu \Delta L}$$

This is often changed to give the pressure drop as a friction head.

The friction head for a length L is found from $h_f = \Delta p/\rho g$

$$h_f = \frac{32\mu L u_m}{\rho g D^2}$$

This is Poiseuille's equation that applies only to laminar flow.

WORKED EXAMPLE 2.2

A capillary tube is 30 mm long and 1 mm bore. The head required to produce a flow rate of 8 mm^3/s is 30 mm. The fluid density is 800 kg/m^3.
Calculate the dynamic and kinematic viscosity of the oil.

SOLUTION

Rearranging Poiseuille's equation we get

$$\mu = \frac{h_f \rho g D^2}{32 L u_m}$$

$$A = \frac{\pi d^2}{4} = \frac{\pi \text{ x } 1^2}{4} = 0.785 \text{ mm}^2$$

$$u_m = \frac{Q}{A} = \frac{8}{0.785} = 10.18 \text{ mm/s}$$

$$\mu = \frac{0.03 \text{ x } 800 \text{ x } 9.81 \text{ x } 0.001^2}{32 \text{ x } 0.03 \text{ x } 0.01018} = 0.0241 \text{ N s/m or } 24.1 \text{ cP}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.0241}{800} = 30.11 \text{ x } 10^{-6} \text{ m}^2/\text{s or } 30.11 \text{ cSt}$$

WORKED EXAMPLE No.2.3

Oil flows in a pipe 100 mm bore with a Reynolds number of 250. The dynamic viscosity is 0.018 Ns/m^2. The density is 900 kg/m^3.

Determine the pressure drop per metre length, the average velocity and the radius at which it occurs.

SOLUTION

$Re = \rho u_m D/\mu$.

Hence $u_m = Re\ \mu/\ \rho D$

$u_m = (250 \text{ x } 0.018)/(900 \text{ x } 0.1) = 0.05$ m/s

$\Delta p = 32 \mu L\ u_m / D^2$

$\Delta p = 32 \text{ x } 0.018 \text{ x } 1 \text{ x } 0.05/0.1^2$

$\Delta p = 2.88$ Pascals.

$u = \{\Delta p/4L\mu\}(R^2 - r^2)$ which is made equal to the average velocity 0.05 m/s

$0.05 = (2.88/4 \text{ x } 1 \text{ x } 0.018)(0.05^2 - r^2)$

$r = 0.035$ m or 35.3 mm.

2.6. FLOW BETWEEN FLAT PLATES

Consider a small element of fluid moving at velocity u with a length dx and height dy at distance y above a flat surface. The shear stress acting on the element increases by dτ in the y direction and the pressure decreases by dp in the x direction. It was shown earlier that $-\frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$

It is assumed that dp/dx does not vary with y so it may be regarded as a fixed value in the following work.

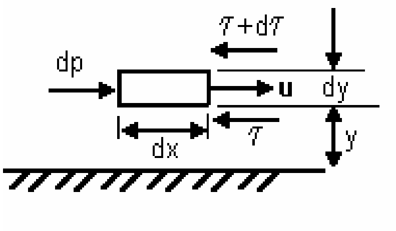


Fig.2.5

Integrating once $-y\frac{\mathrm{dp}}{\mathrm{dx}} = \mu\frac{du}{dy} + A$

Integrating again $-\frac{\mathrm{y}^2}{2}\frac{dp}{dx} = \mu u + Ay + B............(2.6A)$

A and B are constants of integration. The solution of the equation now depends upon the boundary conditions that will yield A and B.

WORKED EXAMPLE No.2.4

Derive the equation linking velocity u and height y at a given point in the x direction when the flow is laminar between two stationary flat parallel plates distance h apart. Go on to derive the volume flow rate and mean velocity.

SOLUTION

When a fluid touches a surface, it sticks to it and moves with it. The velocity at the flat plates is the same as the plates and in this case is zero. The boundary conditions are hence

u = 0 when y = 0

Substituting into equation 2.6A yields that B = 0

u=0 when y=h

Substituting into equation 2.6A yields that A = (dp/dx)h/2

Putting this into equation 2.6A yields

$u = (dp/dx)(1/2\mu)\{y^2 - hy\}$

(The student should do the algebra for this). The result is a parabolic distribution similar that given by Poiseuille's equation earlier only this time it is between two flat parallel surfaces.

FLOW RATE

To find the flow rate we consider flow through a small rectangular slit of width B and height dy at height y.

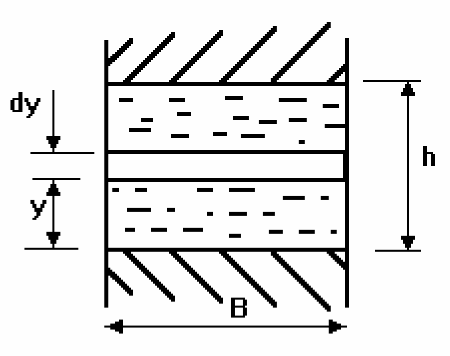


Fig.2.6

The flow through the slit is $dQ = u\,Bdy = (dp/dx)(1/2\mu)\{y^2 - hy\}\,Bdy$

Integrating between y = 0 and y = h to find Q yields

$$\mathbf{Q = -B(dp/dx)(h^3/12\mu)}$$

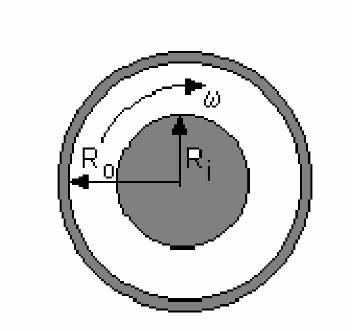
The mean velocity is $u_m = Q/Area = Q/Bh$

hence $\mathbf{u_m = -(dp/dx)(h^2/12\mu)}$

(The student should do the algebra)

2.7 CONCENTRIC CYLINDERS

This could be a shaft rotating in a bush filled with oil or a rotational viscometer. Consider a shaft rotating in a cylinder with the gap between filled with a Newtonian liquid. There is no overall flow rate so equation 2.A does not apply.



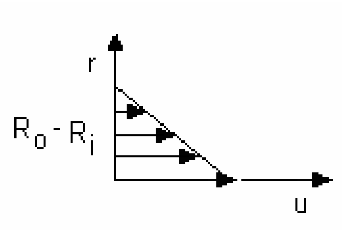


Fig 2.7

Due to the stickiness of the fluid, the liquid sticks to both surfaces and has a velocity $u = \omega R_i$ at the inner layer and zero at the outer layer.

If the gap is small, it may be assumed that the change in the velocity across the gap changes from u to zero linearly with radius r.

$$\tau = \mu \, du/dy$$

But since the change is linear $du/dy = u/(R_o - R_i) = \omega R_i /(R_o - R_i)$

$\tau = \mu \omega R_i /(R_o - R_i)$

Shear force on cylinder F = shear stress x surface area

$$F = 2\pi R_i h \tau = \frac{2\pi R_i^2 h \mu \omega}{R_o - R_i}$$

Torque = F x R_i

$$T = Fr = \frac{2\pi R_i^3 h \mu \omega}{R_o - R_i}$$

In the case of a rotational viscometer we rearrange so that

$$\mu = \frac{T(R_o - R)}{2\pi R_i^3 h \omega}$$

In reality, it is unlikely that the velocity varies linearly with radius and the bottom of the cylinder would have an affect on the torque.

2.8 FALLING SPHERES

This theory may be applied to particle separation in tanks and to a falling sphere viscometer. When a sphere falls, it initially accelerates under the action of gravity. The resistance to motion is due to the shearing of the liquid passing around it. At some point, the resistance balances the force of gravity and the sphere falls at a constant velocity. This is the terminal velocity. For a body immersed in a liquid, the buoyant weight is W and this is equal to the viscous resistance R when the terminal velocity is reached.

R = W = volume x density difference x gravity

$$R = W = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

ρ_s = density of the sphere material

ρ_f = density of fluid

d = sphere diameter

The viscous resistance is much harder to derive from first principles and this will not be attempted here. In general, we use the concept of DRAG and define the DRAG COEFFICIENT as

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}}$$

The dynamic pressure of a flow stream is $\frac{\rho u^2}{2}$

The projected area of a sphere is $\frac{\pi d^2}{4}$

$$C_D = \frac{8R}{\rho u^2 \pi d^2}$$

Research shows the following relationship between C_D and R_e for a sphere.

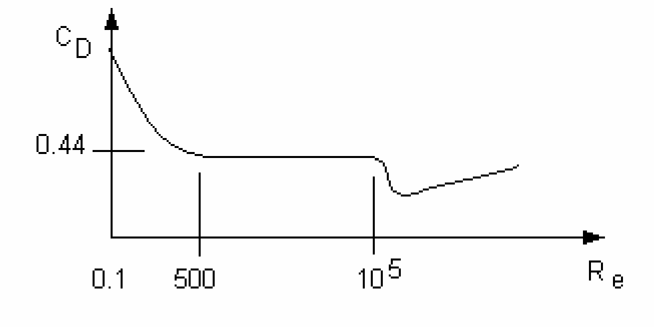


Fig. 2.8

For $R_e<0.2$ the flow is called Stokes flow and Stokes showed that $R = 3\pi d\mu u$ hence $C_D=24\mu/\rho_f ud = 24/R_e$

For $0.2 < R_e < 500$ the flow is called Allen flow and $C_D=18.5R_e^{-0.6}$

For $500 < R_e < 10^5$ C_D is constant $C_D = 0.44$

An empirical formula that covers the range $0.2 < R_e < 10^5$ is as follows.

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

For a falling sphere viscometer, Stokes flow applies. Equating the drag force and the buoyant weight we get

$$3\pi d\mu u = (\pi d^3/6)(\rho_s - \rho_f)\, g$$

$\mu = gd^2(\rho_s - \rho_f)/18u$ for a falling sphere vicometer

The terminal velocity for Stokes flow is $u = d^2g(\rho_s - \rho_f)18\mu$

This formula assumes a fluid of infinite width but in a falling sphere viscometer, the liquid is squeezed between the sphere and the tube walls and additional viscous resistance is produced. The Faxen correction factor F is used to correct the result.

2.9 THRUST BEARINGS

Consider a round flat disc of radius R rotating at angular velocity ω rad/s on top of a flat surface and separated from it by an oil film of thickness t.

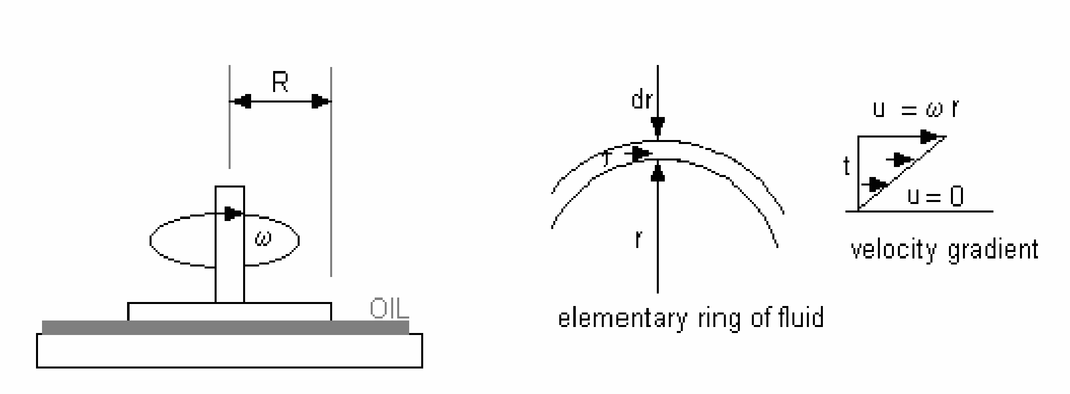


Fig.2.9

Assume the velocity gradient is linear in which case du/dy = u/t = ωr/t at any radius r.

The shear stress on the ring is $\tau = \mu \frac{du}{dy} = \mu \frac{\omega r}{t}$

The shear force is $dF = 2\pi r^2 dr \mu \frac{\omega}{t}$

The torque is $dT = rdF = 2\pi r^3 dr \mu \frac{\omega}{t}$

The total torque is found by integrating with respect to r.

$$T = \int_0^R 2\pi r^3 dr \mu \frac{\omega}{t} = \pi R^4 \mu \frac{\omega}{2t}$$

In terms of diameter D this is $T = \frac{\mu \pi \omega D^4}{32t}$

There are many variations on this theme that you should be prepared to handle.

2.10 MORE ON FLOW THROUGH PIPES

Consider an elementary thin cylindrical layer that makes an element of flow within a pipe. The length is δx , the inside radius is r and the radial thickness is dr. The pressure difference between the ends is δp and the shear stress on the surface increases by dτ from the inner to the outer surface. The velocity at any point is u and the dynamic viscosity is μ.

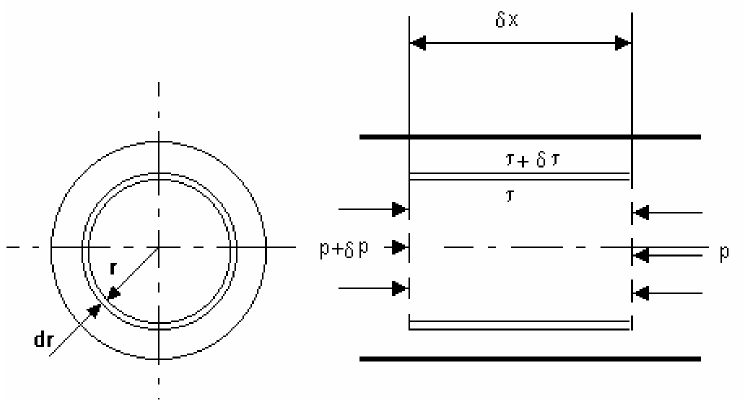


Fig.2.10

The pressure force acting in the direction of flow is $\{\pi(r+dr)^2-\pi r^2\}\delta p$

The shear force opposing is $\{(\tau+\delta\tau)(2\pi)(r+dr) - \tau 2\pi r\}\delta x$

Equating, simplifying and ignoring the product of two small quantities we have the following result.

$$\frac{\delta p}{\delta x} = \frac{\tau}{r} + \frac{d\tau}{dr} \qquad \tau = \mu\frac{du}{dy} \text{ for Newtonian fluids.}$$

If y is measured from the inside of the pipe then r = -y and dy = - dr so $\tau = -\mu\frac{du}{dr}$

$$\frac{\delta p}{\delta x} = -\frac{\mu}{r}\frac{du}{dr} - \mu\frac{d^2u}{dr^2}$$

$$\frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2} = -\frac{1}{\mu}\frac{\delta p}{\delta x}$$

$$\frac{du}{dr} + \frac{rd^2u}{dr^2} = -\frac{r}{\mu}\frac{\delta p}{\delta x}$$

Using partial differentiation to differentiate $\frac{d\left(r\frac{du}{dr}\right)}{dr}$ yields the result $\frac{du}{dr} + \frac{rd^2u}{dr^2}$

hence $\frac{d\left(r\frac{du}{dr}\right)}{dr} = -\frac{r}{\mu}\frac{\delta p}{\delta x}$

Integrating we get $r\frac{du}{dr} = -\frac{r^2}{2\mu}\frac{\delta p}{\delta x} + A$

$$\frac{du}{dr} = -\frac{r}{2\mu}\frac{\delta p}{\delta x} + \frac{A}{r} \quad ....(A)$$

where A is a constant of integration.

Integrating again we get

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + A\ln r + B ....(B)$$

where B is another constant of integration.

Equations (A) and (B) may be used to derive Poiseuille's equation or it may be used to solve flow through an annular passage.

2.10.1 PIPE

At the middle r=0 so from equation (A) it follows that A=0

At the wall, u=0 and r=R. Putting this into equation B yields

$$0 = -\frac{R^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R + B \quad \text{where A = 0}$$

$$B = \frac{R^2}{4\mu}\frac{\delta p}{\delta x}$$

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + \frac{R^2}{4\mu}\frac{\delta p}{\delta x} = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{R^2 - r^2\right\} \text{ and this isPoiseuille's equation again.}$$

2.10.2 ANNULUS

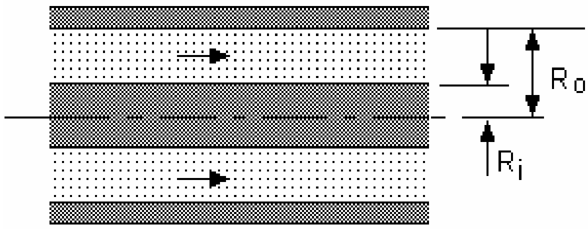


Fig.2.11

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + A\ln r + B$$

The boundary conditions are u = 0 at r = R_i and r = R_o.

$$0 = -\frac{R_o^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R_o + B ....(C)$$

$$0 = -\frac{R_i^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R_i + B ....(D)$$

subtract D from C

$$0 = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{-R_o^2 + R_i^2\right\} + A\left\{\ln R_o - \ln R_i\right\}$$

$$0 = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{R_i^2 - R_0^2\right\} + A\ln\left\{\frac{R_o}{R_i}\right\}$$

$$A = \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}$$

This may be substituted back into equation D. The same result will be obtained from C.

$$0 = -\frac{R_i^2}{4\mu}\frac{\delta p}{\delta x} + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i + B$$

$$B = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[R_i^2 - \left\{\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right]$$

This is put into equation B

$$u = \frac{-r^2}{4\mu}\frac{\delta p}{\delta x} + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[R_i^2 - \left\{\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right]$$

$$u = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[-r^2 + \frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + R_i^2 - \frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i\right]$$

$$u = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[\frac{\left\{R_o^2 - R_i^2\right\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln\frac{r}{R_i} + R_i^2 - r^2\right]$$

For given values the velocity distribution is similar to this.

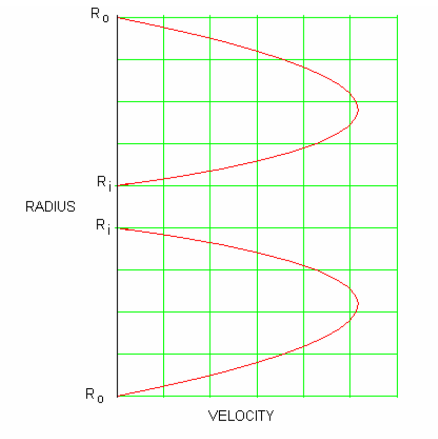


Fig. 2.12

ASSIGNMENT 2

1. Oil flows in a pipe 80 mm bore diameter with a mean velocity of 0.4 m/s. The density is 890 kg/m^3 and the viscosity is 0.075 Ns/m^2. Show that the flow is laminar and hence deduce the pressure loss per metre length.
 (150 Pa per metre).

2. Oil flows in a pipe 100 mm bore diameter with a Reynolds' Number of 500. The density is 800 kg/m^3. Calculate the velocity of a streamline at a radius of 40 mm. The viscosity μ = 0.08 Ns/m^2. (0.36 m/s)

3. A liquid of dynamic viscosity 5 x 10^{-3} Ns/m^2 flows through a capillary of diameter 3.0 mm under a pressure gradient of 1800 N/m^3. Evaluate the volumetric flow rate, the mean velocity, the centre line velocity and the radial position at which the velocity is equal to the mean velocity.
(u_{av} = 0.101 m/s, u_{max} = 0.202 m/s r = 1.06 mm)

4. Similar to Q6 1998

a. Explain the term Stokes flow and terminal velocity.

b. Show that a spherical particle with Stokes flow has a terminal velocity given by
$u = d^2g(\rho_s - \rho_f)/18\mu$
Go on to show that $C_D=24/R_e$

c. For spherical particles, a useful empirical formula relating the drag coefficient and the Reynold's number is

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

Given ρ_f = 1000 kg/m^3, μ= 1 cP and ρ_s= 2630 kg/m^3 determine the maximum size of spherical particles that will be lifted upwards by a vertical stream of water moving at 1 m/s.

d. If the water velocity is reduced to 0.5 m/s, show that particles with a diameter of less than 5.95 mm will fall downwards.

5. Similar to Q5 1998
   A simple fluid coupling consists of two parallel round discs of radius R separated by a a gap h. One disc is connected to the input shaft and rotates at ω_1 rad/s. The other disc is connected to the output shaft and rotates at ω_2 rad/s. The discs are separated by oil of dynamic viscosity μ and it may be assumed that the velocity gradient is linear at all radii.

   Show that the Torque at the input shaft is given by $T = \dfrac{\pi D^4 \mu(\omega_1 - \omega_2)}{32h}$

   The input shaft rotates at 900 rev/min and transmits 500W of power. Calculate the output speed, torque and power. (747 rev/min, 5.3 Nm and 414 W)
   Show by application of max/min theory that the output speed is half the input speed when maximum output power is obtained.

6. Show that for fully developed laminar flow of a fluid of viscosity μ between horizontal parallel plates a distance h apart, the mean velocity u_m is related to the pressure gradient dp/dx by $u_m = -(h^2/12\mu)(dp/dx)$

   Fig.2.11 shows a flanged pipe joint of internal diameter d_i containing viscous fluid of viscosity μ at gauge pressure p. The flange has an outer diameter d_o and is imperfectly tightened so that there is a narrow gap of thickness h. Obtain an expression for the leakage rate of the fluid through the flange.

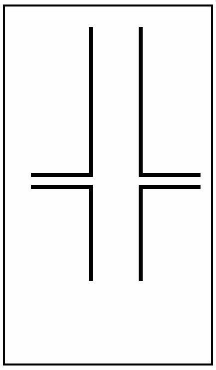


Fig.2.13

   Note that this is a radial flow problem and B in the notes becomes $2\pi r$ and dp/dx becomes -dp/dr. An integration between inner and outer radii will be required to give flow rate Q in terms of pressure drop p.

   The answer is $Q = (2\pi h^3 p/12\mu)/\{\ln(d_o/d_i)\}$

3. TURBULENT FLOW

3.1 FRICTION COEFFICIENT

The friction coefficient is a convenient idea that can be used to calculate the pressure drop in a pipe. It is defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}}$$

3.1.1 DYNAMIC PRESSURE

Consider a fluid flowing with mean velocity u_m. If the kinetic energy of the fluid is converted into flow or fluid energy, the pressure would increase. The pressure rise due to this conversion is called the dynamic pressure.

KE = ½ mu_m^2

Flow Energy = p Q   Q is the volume flow rate and $\rho = m/Q$

Equating   ½ $mu_m^2 = p\,Q$ $p = mu^2/2Q = ½\,\rho\,u_m^2$

3.1.2 WALL SHEAR STRESS τ_o

The wall shear stress is the shear stress in the layer of fluid next to the wall of the pipe.

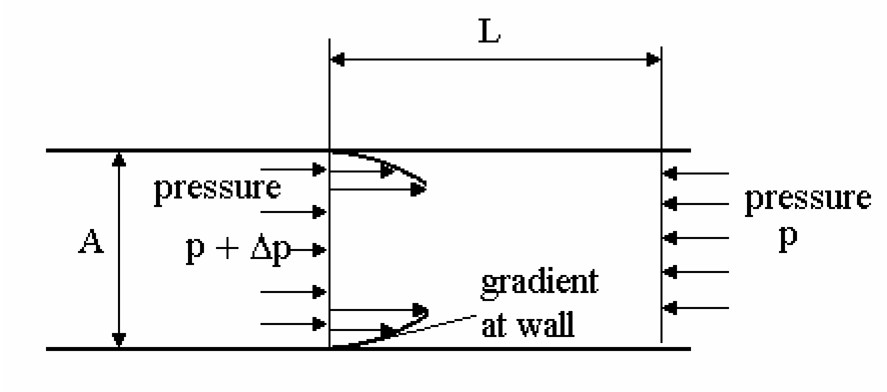


Fig.3.1

The shear stress in the layer next to the wall is $\tau_o = \mu\left(\frac{du}{dy}\right)_{wall}$

The shear force resisting flow is $F_s = \tau_o \pi L D$

The resulting pressure drop produces a force of $F_p = \frac{\Delta p \pi D^2}{4}$

Equating forces gives $\tau_o = \frac{D\Delta p}{4L}$

3.1.3 FRICTION COEFFICIENT for LAMINAR FLOW

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

From Poiseuille's equation $\Delta p = \frac{32\mu L u_m}{D^2}$ Hence $C_f = \left(\frac{2D}{4L\rho u_m^2}\right)\left(\frac{32\mu L u}{D^2}\right) = \frac{16\mu}{\rho u_m^2 D} = \frac{16}{R_e}$

3.1.4 DARCY FORMULA

This formula is mainly used for calculating the pressure loss in a pipe due to turbulent flow but it can be used for laminar flow also.

Turbulent flow in pipes occurs when the Reynolds Number exceeds 2500 but this is not a clear point so 3000 is used to be sure. In order to calculate the frictional losses we use the concept of friction coefficient symbol C_f. This was defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

Rearranging equation to make Δp the subject

$$\Delta p = \frac{4C_f L\rho u_m^2}{2D}$$

This is often expressed as a friction head h_f

$$h_f = \frac{\Delta p}{\rho g} = \frac{4C_f L u_m^2}{2gD}$$

This is the Darcy formula. In the case of laminar flow, Darcy's and Poiseuille's equations must give the same result so equating them gives

$$\frac{4C_f L u_m^2}{2gD} = \frac{32\mu L u_m}{\rho g D^2}$$

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_e}$$

This is the same result as before for laminar flow.

Turbulent flow may be safely assumed in pipes when the Reynolds' Number exceeds 3000. In order to calculate the frictional losses we use the concept of friction coefficient symbol C_f. Note that in older textbooks C_f was written as f but now the symbol f represents $4C_f$.

3.1.5 FLUID RESISTANCE

Fluid resistance is an alternative approach to solving problems involving losses. The above equations may be expressed in terms of flow rate Q by substituting u = Q/A

$$h_f = \frac{4C_f L u_m^2}{2gD} = \frac{4C_f L Q^2}{2gDA^2}$$

Substituting $A = \pi D^2/4$ we get the following.

$$h_f = \frac{32C_f L Q^2}{g\pi^2 D^5} = RQ^2$$

R is the fluid resistance or restriction. $R = \frac{32C_f L}{g\pi^2 D^5}$

If we want pressure loss instead of head loss the equations are as follows.

$$p_f = \rho g h_f = \frac{32\rho C_f L Q^2}{\pi^2 D^5} = RQ^2 \qquad \text{R is the fluid resistance or restriction. } R = \frac{32\rho C_f L}{\pi^2 D^5}$$

It should be noted that R contains the friction coefficient and this is a variable with velocity and surface roughness so R should be used with care.

3.2 MOODY DIAGRAM AND RELATIVE SURFACE ROUGHNESS

In general the friction head is some function of u_m such that $h_f = \phi u_m^n$. Clearly for laminar flow, n =1 but for turbulent flow n is between 1 and 2 and its precise value depends upon the roughness of the pipe surface. Surface roughness promotes turbulence and the effect is shown in the following work.

Relative surface roughness is defined as $\varepsilon = k/D$ where k is the mean surface roughness and D the bore diameter.

An American Engineer called Moody conducted exhaustive experiments and came up with the Moody Chart. The chart is a plot of C_f vertically against R_e horizontally for various values of ε. In order to use this chart you must know two of the three co-ordinates in order to pick out the point on the chart and hence pick out the unknown third co-ordinate. For smooth pipes, (the bottom curve on the diagram), various formulae have been derived such as those by Blasius and Lee.

BLASIUS $C_f = 0.0791\, R_e^{0.25}$

LEE $C_f = 0.0018 + 0.152\, R_e^{0.35}$.

The Moody diagram shows that the friction coefficient reduces with Reynolds number but at a certain point, it becomes constant. When this point is reached, the flow is said to be fully developed turbulent flow. This point occurs at lower Reynolds numbers for rough pipes.

A formula that gives an approximate answer for any surface roughness is that given by Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

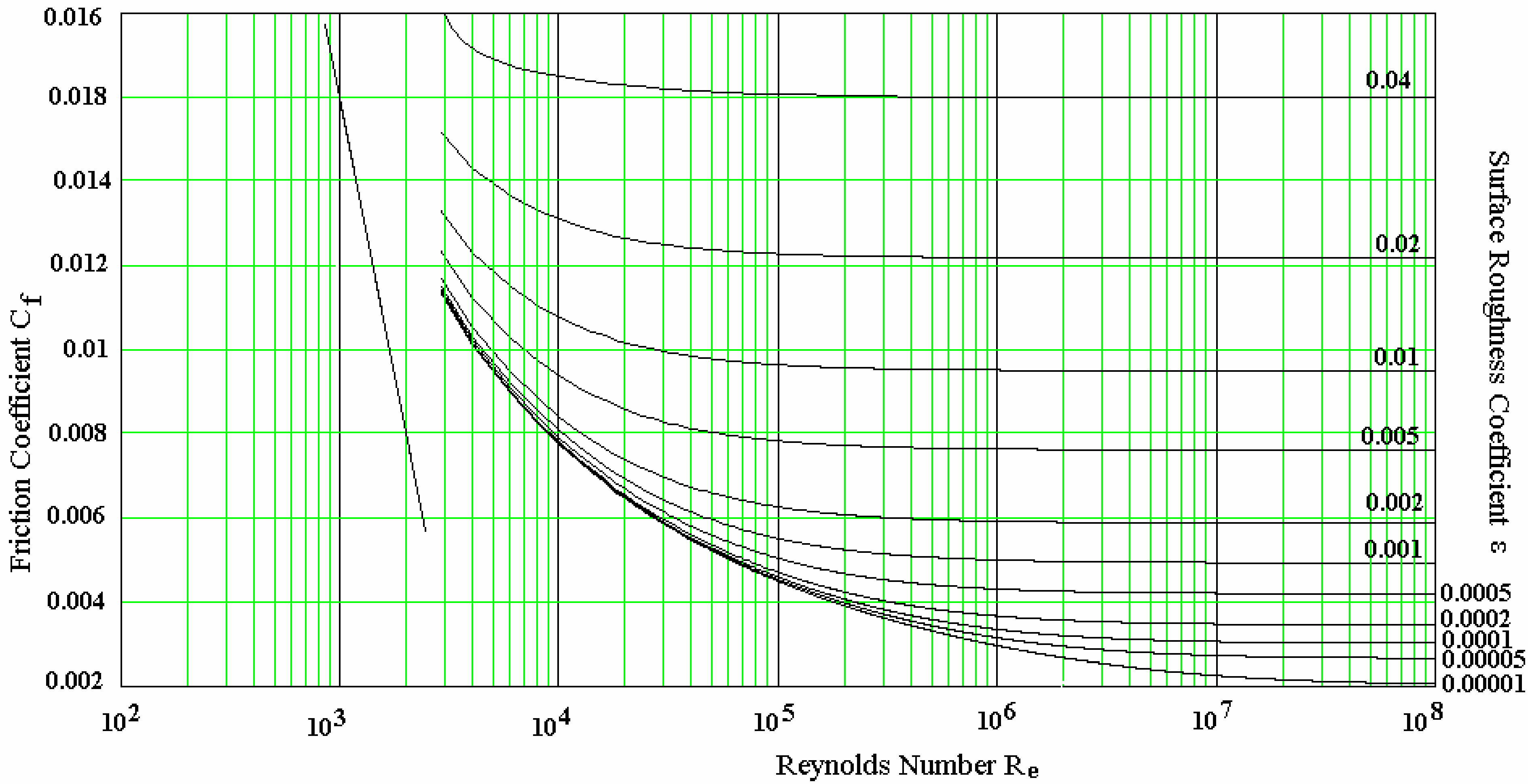

0.016
0.018
0.014
0.012
0.01
0.008
0.006
0.004
0.002
Friction Coefficient C_f
0.04
0.02
0.01
0.005
0.002
0.001
0.0005
0.0002
0.0001
0.00005
0.00001
Surface Roughness Coefficient ε
10^2
10^3
10^4
10^5
10^6
10^7
10^8
Reynolds Number R_e

Fig. 3.2 CHART

WORKED EXAMPLE 3.1

Determine the friction coefficient for a pipe 100 mm bore with a mean surface roughness of 0.06 mm when a fluid flows through it with a Reynolds number of 20 000.

SOLUTION

The mean surface roughness ε = k/d = 0.06/100 = 0.0006
Locate the line for ε = k/d = 0.0006.
Trace the line until it meets the vertical line at Re = 20 000. Read of the value of C_f horizontally on the left. Answer C_f = 0.0067.Check using the formula from Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = 12.206$$

$$C_f = 0.0067$$

WORKED EXAMPLE 3.2

Oil flows in a pipe 80 mm bore with a mean velocity of 4 m/s. The mean surface roughness is 0.02 mm and the length is 60 m. The dynamic viscosity is 0.005 N s/m^2 and the density is 900 kg/m^3. Determine the pressure loss.

SOLUTION

Re = ρud/μ = (900 x 4 x 0.08)/0.005 = 57600

ε= k/d = 0.02/80 = 0.00025

From the chart C_f = 0.0052

h_f = $4C_f$Lu2/2dg = (4 x 0.0052 x 60 x 4^2)/(2 x 9.81 x 0.08) = 12.72 m

Δp = ρgh_f = 900 x 9.81 x 12.72 = 112.32 kPa.

ASSIGNMENT 3

1. A pipe is 25 km long and 80 mm bore diameter. The mean surface roughness is 0.03 mm. It carries oil of density 825 kg/m^3 at a rate of 10 kg/s. The dynamic viscosity is 0.025 N s/m^2.

   Determine the friction coefficient using the Moody Chart and calculate the friction head. (Ans. 3075 m.)

2. Water flows in a pipe at 0.015 m^3/s. The pipe is 50 mm bore diameter. The pressure drop is 13 420 Pa per metre length. The density is 1000 kg/m^3 and the dynamic viscosity is 0.001 N s/m^2.

   Determine
   i. the wall shear stress (167.75 Pa)
   ii. the dynamic pressure (29180 Pa).
   iii. the friction coefficient (0.00575)
   iv. the mean surface roughness (0.0875 mm)

3. Explain briefly what is meant by fully developed laminar flow. The velocity u at any radius r in fully developed laminar flow through a straight horizontal pipe of internal radius r_o is given by

$$u = (1/4\mu)(r_o^2 - r^2)dp/dx$$

   dp/dx is the pressure gradient in the direction of flow and μ is the dynamic viscosity.

   Show that the pressure drop over a length L is given by the following formula.

$$\Delta p = 32\mu L u_m/D^2$$

   The wall skin friction coefficient is defined as $C_f = 2\tau_o/(\rho u_m^2)$.

   Show that $C_f = 16/R_e$ where $R_e = \rho u_m D/\mu$ and ρ is the density, u_m is the mean velocity and τ_o is the wall shear stress.

4. Oil with viscosity 2 x 10^{-2} Ns/m^2 and density 850 kg/m^3 is pumped along a straight horizontal pipe with a flow rate of 5 dm^3/s. The static pressure difference between two tapping points 10 m apart is 80 N/m^2. Assuming laminar flow determine the following.

   i. The pipe diameter.
   ii. The Reynolds number.

   Comment on the validity of the assumption that the flow is laminar

4. NON-NEWTONIAN FLUIDS

A Newtonian fluid as discussed so far in this tutorial is a fluid that obeys the law $\tau = \mu \frac{du}{dy} = \mu\dot{\gamma}$

A Non – Newtonian fluid is generally described by the non-linear law $\tau = \tau_y + k\dot{\gamma}^n$

τ_y is known as the yield shear stress and $\dot{\gamma}$ is the rate of shear strain. Figure 4.1 shows the principle forms of this equation.

Graph A shows an ideal fluid that has no viscosity and hence has no shear stress at any point. This is often used in theoretical models of fluid flow.

Graph B shows a Newtonian Fluid. This is the type of fluid with which this book is mostly concerned, fluids such as water and oil. The graph is hence a straight line and the gradient is the viscosity μ.

There is a range of other liquid or semi-liquid materials that do not obey this law and produce strange flow characteristics. Such materials include various foodstuffs, paints, cements and so on. Many of these are in fact solid particles suspended in a liquid with various concentrations.

Graph C shows the relationship for a ***Dilatent fluid***. The gradient and hence viscosity increases with $\dot{\gamma}$ and such fluids are also called ***shear-thickening.*** This phenomenon occurs with some solutions of sugar and starches.

Graph D shows the relationship for a ***Pseudo-plastic***. The gradient and hence viscosity reduces with $\dot{\gamma}$ and they are called ***shear-thinning.*** Most foodstuffs are like this as well as clay and liquid cement..

Other fluids behave like a ***plastic*** and require a minimum stress before it shears τ_y. This is plastic behaviour but unlike plastics, there may be no elasticity prior to shearing.

Graph E shows the relationship for a ***Bingham plastic.*** This is the special case where the behaviour is the same as a Newtonian fluid except for the existence of the yield stress. Foodstuffs containing high level of fats approximate to this model (butter, margarine, chocolate and Mayonnaise).

Graph F shows the relationship for a ***plastic*** fluid that exhibits shear thickening characteristics.

Graph G shows the relationship for a ***Casson fluid***. This is a plastic fluid that exhibits shear-thinning characteristics. This model was developed for fluids containing rod like solids and is often applied to molten chocolate and blood.

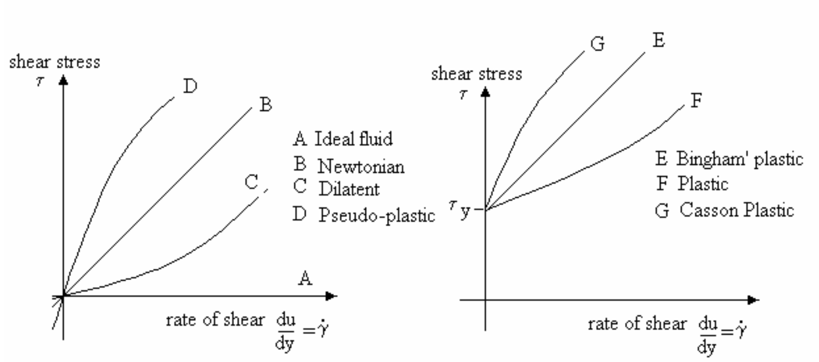


Fig.4.1

MATHEMATICAL MODELS

The graphs that relate shear stress τ and rate of shear strain γ are based on models or equations. Most are mathematical equations created to represent empirical data.

Hirschel and Bulkeley developed the power law for non-Newtonian equations. This is as follows.

$\tau = \tau_y + K\dot{\gamma}^n$ K is called the consistency coefficient and n is a power.

In the case of a Newtonian fluid n = 1 and $\tau_y = 0$ and K = μ (the dynamic viscosity) $\tau = \mu\dot{\gamma}$

For a Bingham plastic, n = 1 and K is also called the plastic viscosity μ_p. The relationship reduces to

$$\tau = \tau_y + \mu_p\dot{\gamma}$$

For a ***dilatent fluid,*** $\tau_y = 0$ and $n>1$

For a ***pseudo-plastic,*** $\tau_y = 0$ and $n<1$

The model for both is $\tau = K\dot{\gamma}^n$

The ***Herchel-Bulkeley*** model is as follows. $\tau = \tau_y + K\dot{\gamma}^n$

This may be developed as follows.

$$\tau = \tau_y + K\dot{\gamma}^n$$

$\tau - \tau_y = K\dot{\gamma}^n$ sometimes written as $\tau - \tau_y = \mu_p\dot{\gamma}^n$ where μ_p is called the plastic viscosity.

dividing by $\dot{\gamma}$

$$\frac{\tau}{\dot{\gamma}} - \frac{\tau_y}{\dot{\gamma}} = K\frac{\dot{\gamma}^n}{\dot{\gamma}} = K\dot{\gamma}^{n-1}$$

$\frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$ The ratio is called the apparent viscosity μ_{app}

$$\mu_{app} = \frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$$

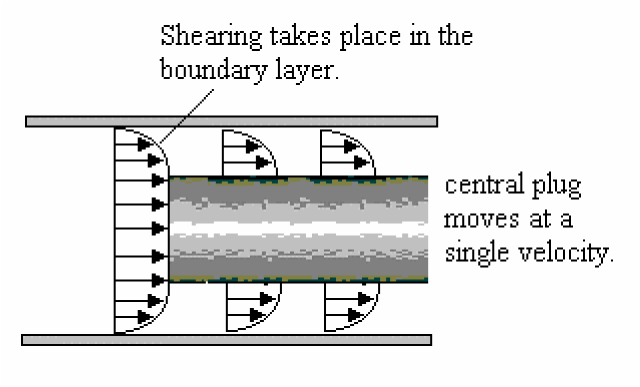
For a Bingham plastic $n = 1$ so $\mu_{app} = \frac{\tau_y}{\dot{\gamma}} + K$

For a Fluid with no yield shear value $\tau_y = 0$ so $\mu_{app} = K\dot{\gamma}^{n-1}$

The ***Casson*** fluid model is quite different in form from the others and is as follows.

$$\tau^{\frac{1}{2}} = \tau_y^{\frac{1}{2}} + K\dot{\gamma}^{\frac{1}{2}}$$

THE FLOW OF A PLASTIC FLUID



Note that fluids with a shear yield stress will flow in a pipe as a plug. Within a certain radius, the shear stress will be insufficient to produce shearing so inside that radius the fluid flows as a solid plug. Fig. 4.2 shows a typical situation for a Bingham Plastic.

Fig.4.2

MINIMUM PRESSURE

The shear stress acting on the surface of the plug is the yield value. Let the plug be diameter d.
The pressure force acting on the plug is $\Delta p \times \pi d^2/4$
The shear force acting on the surface of the plug is $\tau_y \times \pi\, d\, L$
Equating we find $\Delta p \times \pi d^2/4 = \tau_y \times \pi\, d\, L$

$d = \tau_y \times 4\, L/\Delta p$ or $\Delta p = \tau_y \times 4\, L/d$

The minimum pressure required to produce flow must occur when d is largest and equal to the bore of the pipe. $\Delta p\ (\text{minimum}) = \tau_y \times 4\, L/D$

The diameter of the plug at any greater pressure must be given by $d = \tau_y \times 4\, L/\Delta p$

For a Bingham Plastic, the boundary layer between the plug and the wall must be laminar and the velocity must be related to radius by the formula derived earlier.

$$u = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) = \frac{\Delta p}{16\mu L}\left(D^2 - d^2\right)$$

FLOW RATE

The flow rate should be calculated in two stages. The plug moves at a constant velocity so the flow rate for the plug is simply $Q_p = u \times \text{cross sectional area} = u \times \pi d^2/4$

The flow within the boundary layer is found in the usual way as follows. Consider an elementary ring radius r and width dr.

$$dQ = u \times 2\pi r\, dr = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) \times 2\pi r\, dr$$

$$Q = \frac{\Delta p \pi}{2\mu L}\int_R^r \left(rR^2 - r^3\right) dr$$

$$Q = \frac{\Delta p \pi}{2\mu L}\left[\frac{r^2R^2}{2} - \frac{r^4}{4}\right]_r^R = \frac{\Delta p \pi}{2\mu L}\left[\left(\frac{R^4}{2} - \frac{R^4}{4}\right) - \left(\frac{r^2R^2}{2} - \frac{r^4}{4}\right)\right]$$

$$Q = \frac{\Delta p \pi}{2\mu L}\left[\left(\frac{R^4}{4}\right) - \frac{r^2R^2}{2} + \frac{r^4}{4}\right]$$

The mean velocity as always is defined as u_m = Q/Cross sectional area.

WORKED EXAMPLE 4.1

The Herchel-Bulkeley model for a non-Newtonian fluid is as follows. $\tau = \tau_y + K\dot{\gamma}^n$.

Derive an equation for the minimum pressure required drop per metre length in a straight horizontal pipe that will produce flow.

Given that the pressure drop per metre length in the pipe is 60 Pa/m and the yield shear stress is 0.2 Pa, calculate the radius of the slug sliding through the middle.

SOLUTION

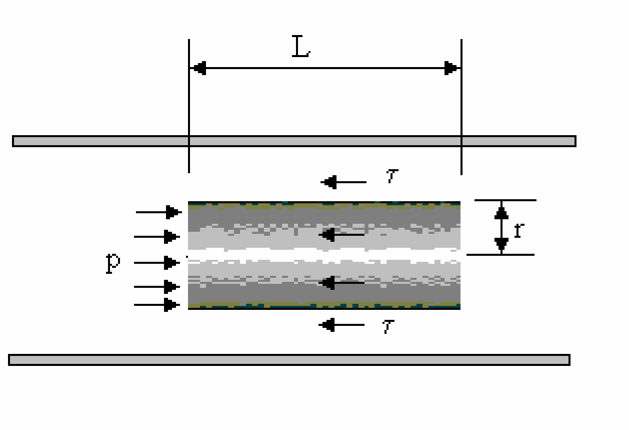


Fig. 3.3

The pressure difference p acting on the cross sectional area must produce sufficient force to overcome the shear stress τ acting on the surface area of the cylindrical slug. For the slug to move, the shear stress must be at least equal to the yield value τy. Balancing the forces gives the following.

$p \times \pi r^2 = \tau_y \times 2\pi r L$

$p/L = 2\tau_y / r$

$60 = 2 \times 0.2/r$

$r = 0.4/60 = 0.0066$ m or 6.6 mm

WORKED EXAMPLE 4.2

A Bingham plastic flows in a pipe and it is observed that the central plug is 30 mm diameter when the pressure drop is 100 Pa/m.

Calculate the yield shear stress.

Given that at a larger radius the rate of shear strain is 20 s^{-1} and the consistency coefficient is 0.6 Pa s, calculate the shear stress.

SOLUTION

For a Bingham plastic, the same theory as in the last example applies.

$p/L = 2\tau_y /r$
$100 = 2\ \tau_y/0.015$
$\tau_y = 100 \times 0.015/2 = 0.75$ Pa

A mathematical model for a Bingham plastic is

$$\tau = \tau_y + K\dot{\gamma} = 0.75 + 0.6 \times 20 = 12.75 \text{ Pa}$$

ASSIGNMENT 4

1. Research has shown that tomato ketchup has the following viscous properties at 25°C.

   Consistency coefficient K = 18.7 Pa s^n
   Power n = 0.27
   Shear yield stress = 32 Pa

   Calculate the apparent viscosity when the rate of shear is 1, 10, 100 and 1000 s^{-1} and conclude on the effect of the shear rate on the apparent viscosity.

   Answers

   $\gamma = 1$ $\mu_{app} = 50.7$
   $\gamma = 10$ $\mu_{app} = 6.682$
   $\gamma = 100$ $\mu_{app} = 0.968$
   $\gamma = 1000$ $\mu_{app} = 0.153$

2. A Bingham plastic fluid has a viscosity of 0.05 N s/m^2 and yield stress of 0.6 N/m^2. It flows in a tube 15 mm bore diameter and 3 m long.

   (i) Evaluate the minimum pressure drop required to produce flow. (480 N/m^2)

   The actual pressure drop is twice the minimum value. Sketch the velocity profile and calculate the following.

   (ii) The radius of the solid core. (3.75 mm)
   (iii) The velocity of the core. (67.5 mm/s)
   (iv) The volumetric flow rate. (7.46 cm^3/s)

3. A non-Newtonian fluid is modelled by the equation $\tau = K\left(\frac{du}{dr}\right)^n$ where n = 0.8 and K = 0.05 N $s^{0.8}$/m^2. It flows through a tube 6 mm bore diameter under the influence of a pressure drop of 6400 N/m^2 per metre length. Obtain an expression for the velocity profile and evaluate the following.

   (i) The centre line velocity. (0.953 m/s)
   (ii) The mean velocity. (0.5 m/s)