Characteristic strength, Weibull modulus, and failure probability of fused silica glass

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Abstract. The development of high-energy lasers has focused attention on the requirement to assess the mechanical strength of optical components made of fused silica or fused quartz (SiO_2). The strength of this material is known to be highly dependent on the stressed area and the surface finish, but has not yet been properly characterized in the published literature. Recently, Detrio and collaborators at the University of Dayton Research Institute (UDRI) performed extensive ring-on-ring flexural strength measurements on fused SiO_2 specimens ranging in size from 1 to 9 in. in diameter and of widely differing surface qualities. We report on a Weibull statistical analysis of the UDRI data—an analysis based on the procedure outlined in *Proc. SPIE* **4375**, 241 (2001). We demonstrate that (1) a two-parameter Weibull model, including the area-scaling principle, applies; (2) the shape parameter ($m \simeq 10$) is essentially independent of the stressed area as well as the surface finish; and (3) the characteristic strength (1-cm^2 uniformly stressed area) obeys a linear law, σ_C (in megapascals) $\simeq 160 - 2.83 \times \overline{\text{PBS}}$ (in parts per million per steradian), where $\overline{\text{PBS}}$ characterizes the surface/subsurface "damage" of an appropriate set of test specimens. In this light, we evaluate the cumulative failure probability and the failure probability density of polished and superpolished fused SiO_2 windows as a function of the biaxial tensile stress, for stressed areas ranging from 0.3 to 100 cm^2. *© 2009 Society of Photo-Optical Instrumentation Engineers.* [DOI: 10.1117/1.3265716]

Subject terms: failure probability; flexural strength; fused silica; stressed area; surface finish; Weibull statistics.

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1 Introduction

Fused SiO_2 glasses (fused quartz and fused silica) are materials of much value in many technological applications. In addition to outstanding optical properties, fused SiO_2 has a very low coefficient of thermal expansion, which makes it highly attractive for designing optical components such as laser windows.[1] For this reason, it is essential to thoroughly investigate key features relating to the flexural strength, especially how stressed area and surface finish impact the mechanical performance. Readily available literature does not go beyond reporting mean strength values (or modulus of rupture) of 50 MPa (Ref. 2)—or 110 MPa (Ref. 3)—and a Weibull shape parameter of 4 to 5 (Ref. 4). Very recently, this situation has drastically improved in view of a comprehensive investigation that was carried out at the University of Dayton Research Institute[5] (UDRI); this investigation includes rich sets of fracture-stress measurements performed in conjunction with detailed assessments of the surface finish of tested specimens. In this paper, we document the results of an analysis of the UDRI data—an analysis based on Weibull's theory of brittle fracture[6]—and clarify issues relating to the failure probability dependence on biaxial tensile stresses, taking into consideration the stressed area as well as the surface finish.

The specimens tested at UDRI were disks made of polished GE type-124 fused quartz[7,*] having diameters of 1, 3, and 9 in.; in addition, some 1-in.-diam disks were "superpolished" for the specific purpose of investigating how removing residual surface/subsurface flaws may enhance the fracture strength. Each specimen was tested to fracture in a ring-on-ring load fixture with innerring sizes adjusted to match the diameter of the test specimen. Such test configurations generate equibiaxial tensile stresses of uniform intensity over the opposite surface area delineated by the inner ring and, therefore, are readily amenable to a Weibull statistical analysis of the fracture-inducing stresses. Furthermore, work performed at UDRI includes comprehensive evaluations of the surface/subsurface quality of many of the fracture-tested specimens. This was done by means of PBS® measurements, which rely on a narrow helium-

[*] There are two distinct processes currently used to produce amorphous SiO_2 glass: thermal fusion of crystalline quartz (fused quartz) and chemical synthesis from high-purity precursor silicon tetrachloride (fused silica). The two processes yield products of different impurity content but identical elastic and mechanical properties.

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neon laser beam to probe the extent of damage through mapping of the intensity of the *P*-polarized backscattered light.[8] In Sec. 4, it is shown that the PBS method provides an essential tool for interpreting the results of fracture testing and, by the same token, to confirm that the surface quality of an SiO_2 glass plays a major role in controlling the fracture-initiating process.[9]

Section 2 addresses the issue of properly interpreting the results of fracture-strength measurements performed on brittle materials in a concentric-ring test configuration; specifically, we formulate Weibull's theory assuming that (1) the applied stress is uniform over a specified area and (2) the failure originates from stress-concentrating flaws located at or near the surface. This model leads to the concept of a characteristic strength σ_C, which defines the strength of a material subjected to uniform loadings over a 1-cm^2 area. In this context, the results of testing that was carried out at UDRI and listed in Table 3 of Ref. 5 can be evaluated in a straightforward manner, thereby yielding reliable numbers for characterizing the inherent strength of an SiO_2 glass (Sec. 3). As mentioned earlier, in Sec. 4 we examine the dependence of σ_C on the surface finish as assessed by means of PBS measurements. In conjunction with shape-parameter (Weibull modulus) values obtained through fitting the cumulative failure probability as well as the failure probability density, this enables us to describe the failure probability features of an SiO_2 glass, for stressed areas ranging from less than 1 to 100 cm^2, taking the surface finish into consideration (Sec. 5). The conclusions are stated in Sec. 6; in addition, the Appendix addresses issues relating to the interpretation of PBS measurements in relation to recorded fracture-stress data.

2 Weibull Model

The results of fracture testing are usually reported in terms of an average strength,

$$\overline{\sigma_i} = 1/n\sum_{i=1}^{n} \sigma_i \pm \Delta\sigma_i, \tag{1}$$

i.e., in terms of a mean value (± standard deviation) of the recorded stresses at failure. This "strength" does not represent an objective measure of the inherent strength since $\overline{\sigma_i}$ depends on the test method in addition to the size of the volume or the area subjected to tensile stresses.[10] Furthermore, it is common practice to interpret the results of fracture testing on the basis of the distribution

$$P(\sigma) = 1 - \exp[-(\sigma/\sigma_N)^m], \tag{2}$$

where $P(\sigma)$ denotes the cumulative failure probability as a function of the applied tensile stress. This expression involves two parameters—the nominal strength σ_N and the shape parameter m—and both can be extracted from a set of experimental data by fitting the estimated failure probability to Eq. (2). The usual procedure for performing this task consists of obtaining a least-squares fit to a linearized version of Eq. (2):

$$\ln\{-\ln[1 - P(\sigma)]\} = -m\ln(\sigma_N) + m\ln(\sigma), \tag{3}$$

which yields an approximate number for the shape parameter and a "good" number for the nominal strength (cumulative failure probability of 63%). Evidently, the strength σ_N does not take into account the potential impact of the test method and, therefore, does not relate to the inherent strength in an obvious manner.

For our purposes, and keeping in mind that fracture of brittle materials subjected to tensile stresses occurs as a result of stress enhancements generated by randomly distributed surface/subsurface imperfections, we recall that according to Weibull's two-parameter model, that is, if fracture can occur at any level of applied stress, the cumulative failure probability of a test specimen subjected to a stress distribution $\sigma(x,y)$ on the surface under tension can be expressed as follows:

$$P = 1 - \exp\left\{-\int_{\text{surf}}\left[\frac{\sigma(x,y)}{\chi}\right]^m \mathrm{d}x\,\mathrm{d}y\right\}. \tag{4}$$

The scaling parameter χ and the shape parameter m are properties of the material under consideration and, therefore, independent of the testing procedure or the specimen size. In a concentric-ring test configuration, the equibiaxial stresses acting on the surface under tension are effectively uniform over the entire gauge area S, which implies that Eq. (4) reduces to

$$P(\sigma) = 1 - \exp\left[-S\left(\frac{\sigma}{\chi}\right)^m\right], \tag{5}$$

thus emphasizing the impact of the stressed area on the failure probability; similarly, the failure-probability density obeys the function

$$\frac{\mathrm{d}P(\sigma)}{\mathrm{d}\sigma} = S\frac{m\sigma^{m-1}}{\chi^m}\exp\left[-S\left(\frac{\sigma}{\chi}\right)^m\right], \tag{6}$$

which describes the distribution of failure stresses, i.e., the probability of a failure occurring at the stress level σ.

At this point, we can attempt to relate the scaling parameter to the effective strength σ_E, that is, the expected mean stress at failure in a given test environment. For this purpose we write

$$\bar{\sigma} = \int_0^\infty \sigma\left[\frac{\mathrm{d}P(\sigma)}{\mathrm{d}\sigma}\right]\mathrm{d}\sigma, \tag{7}$$

with $\mathrm{d}P/\mathrm{d}\sigma$ as in Eq. (6), which yields

$$\sigma_E = \frac{\chi}{S^{1/m}}\Gamma\left(1 + \frac{1}{m}\right), \tag{8}$$

where $\Gamma(z)$ designates the gamma factorial function. Suppose now that the stressed surface has an area s equal to one unit, say 1 cm^2; in that case, Eq. (7) yields the characteristic strength,

Table 1 Weibull statistical analysis of fused SiO_2 fracture-test data generated at UDRI.

	1-in. diam.	3-in. diam	9-in. diam.	1-in. Repolished
Number of specimens (1)	28	25	23	11
Stressed area (cm^2)	0.897	6.41	71.2	0.503
Measured strength[a] (MPa)	109±14	102±11	77.7±13.2	172±20
Derived from Weibull plots				
Shape parameter[b] (1)	8.82±0.79	10.6±0.6	6.08±0.85	10.2±1.9
Nominal strength[b] (MPa)	115±2	107±1	83.9±2.5	180±4
Derived from $P(\sigma)$ versus σ				
Shape parameter[b] (1)	11.7±1.5	10.2±0.9	8.37±0.87	9.83±1.62
Characteristic strength[b] (MPa)	109±1	123±2	131±7	159±4
Effective strength[c] (MPa)	~110	~103	~78.7	~171
Derived from $dP/d\sigma$ versus σ				
Shape parameter[b] (1)	13.6±4.5	10.4±3.4	8.46±3.43	N/A
Characteristic strength[b] (MPa)	111±3	123±6	131±24	N/A
Effective strength[c] (MPa)	~112	~103	~79.1	N/A

[a]Average±standard deviation.
[b]At the 95% confidence level.
[c]Based on the area-scaling law.
N/A=not available.

$$\sigma_C = \frac{\chi}{s^{1/m}}\Gamma\left(1+\frac{1}{m}\right), \tag{9}$$

thus demonstrating that the area-scaling law can be expressed as follows:

$$\sigma_E = \frac{\sigma_C}{(S/s)^{1/m}}, \tag{10}$$

which tells us how to estimate the effective strength if the parameters σ_C and m are available.

Returning now to Eq. (5), and taking Eq. (9) into account, it is immediately seen that on eliminating the scaling parameter χ, the cumulative failure probability can be expressed in a convenient and explicit form:

$$P(\sigma) = 1 - \exp\left\{-\frac{S}{s}\left[\Gamma\left(1+\frac{1}{m}\right)\right]^m\left(\frac{\sigma}{\sigma_C}\right)^m\right\}, \tag{11}$$

on assuming that the area S is subjected to uniform stresses. The failure probability density, therefore, relates to stressed area and material properties as follows:

$$\frac{dP(\sigma)}{d\sigma} = \frac{S}{s}\frac{m\sigma^{m-1}}{(\sigma_C)^m}\left[\Gamma\left(1+\frac{1}{m}\right)\right]^m \times\exp\left\{-\frac{S}{s}\left[\Gamma\left(1+\frac{1}{m}\right)\right]^m\left(\frac{\sigma}{\sigma_C}\right)^m\right\}. \tag{12}$$

In the next section, we take advantage of this expression to describe the dispersion of failure stresses and, by the same token, the distribution of critical flaw sizes on the surface of polished SiO_2 glass.

3 Data Analysis

Our analysis of the flexural strength data recorded in Ref. 5 amounts to obtaining the parameters σ_C and m, which control the failure probability of test specimens originating from the same lot and having the same surface finish. The results, including key relevant numbers, are listed in Table 1. First, note that experimentation conducted at UDRI includes three lots of "as-received" specimens measuring 1, 3, and 9 in. in diameter; in addition, fracture-stress measurements were performed on a smaller number of "superpolished" 1-in.-diam specimens. Depending on the size of the specimens, Detrio et al.[5] used different sets of support and loading rings, which implies substantially different equibiaxially stressed gauge areas, as specified in Table 1. The measured strengths—as defined in Eq. (1)—clearly

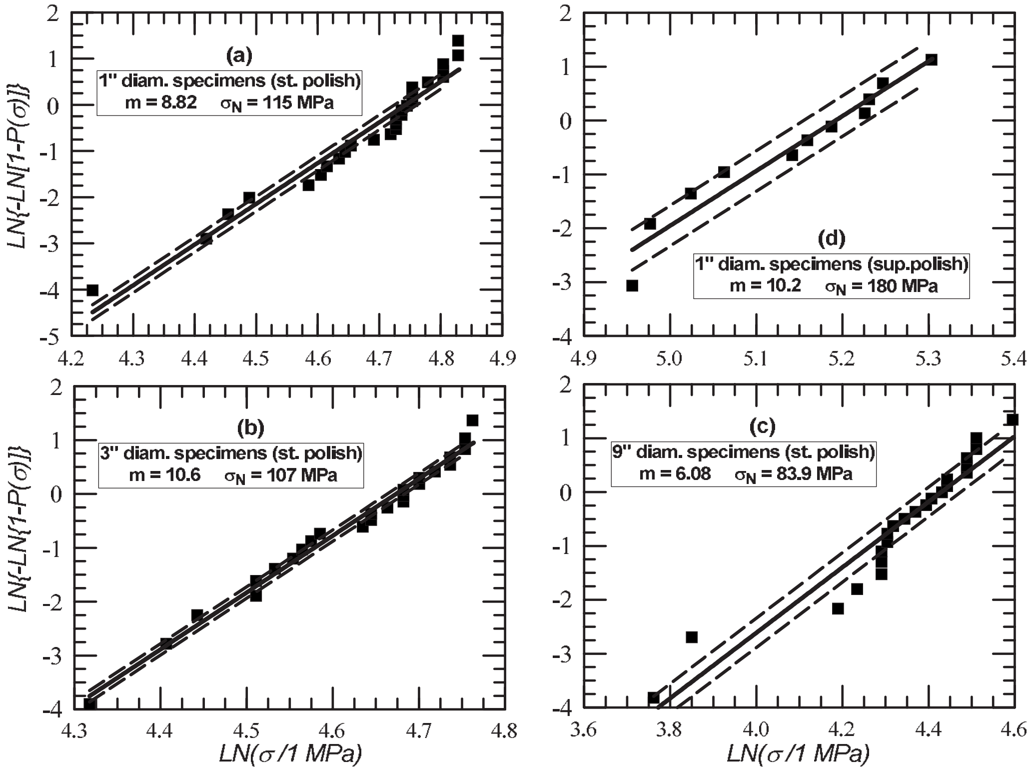

Fig. 1 Fused SiO_2 Weibull plots based on flexural strength data as listed in Ref. 5. The broken lines are indicative of the 95% confidence bands. Parameter-value uncertainties are recorded in Table 1.

show that the strength of fused SiO_2 glass decreases as the area subjected to stresses increases and, furthermore, that repolishing substantially enhances the strength. The procedure we have adopted to interpret the data consists of three steps as outlined next.

3.1 *Step 1*

The initial task consists of estimating the cumulative failure probability from available failure-stress data; this is usually done by ranking the failure stresses (σ_i) in ascending order ($i=1,2,\ldots,n$) and assigning probabilities of failure according to $P_i=(i-0.5)/n$, where n is the number of broken specimens (as recommended in Ref. 11).[†] On fitting the $\ln[-\ln(1-P_i)]$ versus $\ln(\sigma_i)$ data points to a straight line [see Eq. (3)], the procedure provides not only a visual assessment of the validity of a two-parameter model but also a direct estimate of the shape parameter and the nominal strength. Figure 1 illustrates the methodology based on the totality of applicable data for the four sets of fused SiO_2 specimens, and Table 1 lists the relevant statistical parameter values as derived from Weibull plots. Three comments are in order:

1. With the possible exception of the 9-in.-diam specimens, the recorded fracture stresses obey the two-parameter model exceptionaly well, thus demonstrating the absence of residual stresses and suggesting an essentially unimodal flaw-size dispersion.
2. The nominal strength decreases as the gauge area increases, which—evidently—reflects the probability of exposing the test specimens to a larger strength-limiting defect, thus enhancing the likelihood of failure at lower stress levels.
3. The shape-parameter values derived from the Weibull plots displayed in Fig. 1 cover a fairly broad range ($6\leqslant m\leqslant 11$), but keep in mind that the linearization procedure places inordinate weight on the low-strength data points and may result in incorrect numbers.[13]

3.2 *Step 2*

An obvious way of alleviating issues arising from relying on Weibull plots is to take advantage of available software to obtain reliable parameter values simply through nonlinear fitting of the P_i versus σ_i data points.[14] Figure 2 depicts the situation on fitting the UDRI failure-stress data to the cumulative failure probability expression, as formulated in Eq. (11). The fits are of remarkable quality (CoD$\simeq$0.980) and yield shape-parameter values that cluster around 10, thus substantiating our contention that m values obtained through linearization cannot be relied on. The characteristic strength values (1-cm^2 stressed area) clearly demonstrate the strength enhancement ($\sim$45%) achieved through repolishing, but the numbers for 1-, 3-, and 9-in.-diam specimens are not consistent with Weibull's model, if—as tacitly

[†]Note that there is no agreement regarding the best expression for P_i. Lawn,[12] for instance, writes $P_i=i/(n+1)$, which may have significant implications for poorly populated data sets.

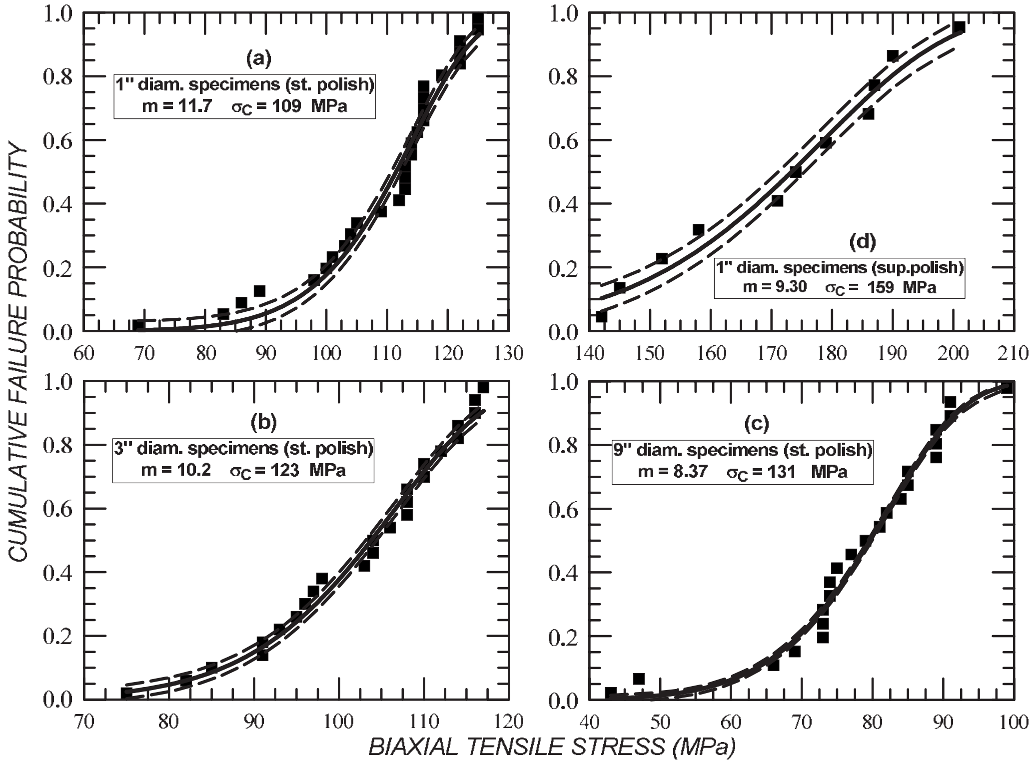

Fig. 2 Cumulative failure probability distributions of the four sets of fused SiO_2 specimens fracture-tested at UDRI. The solid lines illustrate the results of fitting estimated probabilities to Eq. (11). The broken lines are indicative of the 95% confidence bands; parameter-value uncertainties are recorded in Table 1.

assumed in Ref. 5—as-received specimens exhibit identical surface finish features. This issue is addressed in Sec. 4. Returning now to Eq. (10), such characteristic strengths, together with relevant shape parameters, yield effective strength values in excellent agreement with the measured strengths (see Table 1), which validates the methodology we are advocating here.

3.3 *Step 3*

In principle, we can avoid relying on estimated cumulative failure probabilities P_i by fitting the distribution of measured failure stresses σ_i to Eq. (12)—in other words, making use of the Weibull statistical distribution function. To perform this task, we divide the recorded failure stresses into 5-MPa-wide "bins," thus creating the histograms displayed in Fig. 3. Fitting was carried out by means of the Marquardt-Levenberg method on injecting a normalization constant C, which means dealing with three independent parameters (C, m, and σ_C) and, therefore, substantially broader confidence bands compared to fitting cumulative failure-probability distributions. Note that

1. The dispersion of failure stresses is strongly skewed toward abnormally weak specimens (skewness coefficient=−0.586 for as-received 1-in.-diam specimens), which reflects the observation that unusually large flaws—the weakest link—result in abnormally weak specimens, whereas unusually small flaws do not impact the strength.
2. The histograms clearly show that the procedure requires larger data sets than currently available; in effect, fitting the dispersion of the recorded fracture stresses for 11 superpolished 1-in.-diam specimens turned out to be unsuccessful.
3. We see in Table 1 that the results of fitting the dispersions are in surprisingly close agreement with shape parameter as well as characteristic strength values obtained previously (step 2), which further substantiates the claim that these are correct numbers for fused SiO_2 based on UDRI experimentation.

4 Weibull Parameters

The strength properties of a brittle material such as fused SiO_2 glass are controlled not only by the number of surface/subsurface flaws but, more significantly, by their size distribution. In this light, we now examine the shape parameter and the characteristic strength values listed in Table 1. Specifically, we demonstrate that (1) the specimens investigated at UDRI have similar shape parameters, hence similar flaw-size distributions, and (2) the issue arising in connection with the characteristic strength of as-received specimens must be attributed to differences in surface finish, hence different critical flaw populations.

4.1 *Shape Parameter*

The shape-parameter values derived from fitting cumulative failure probabilities through nonlinear regressions (see

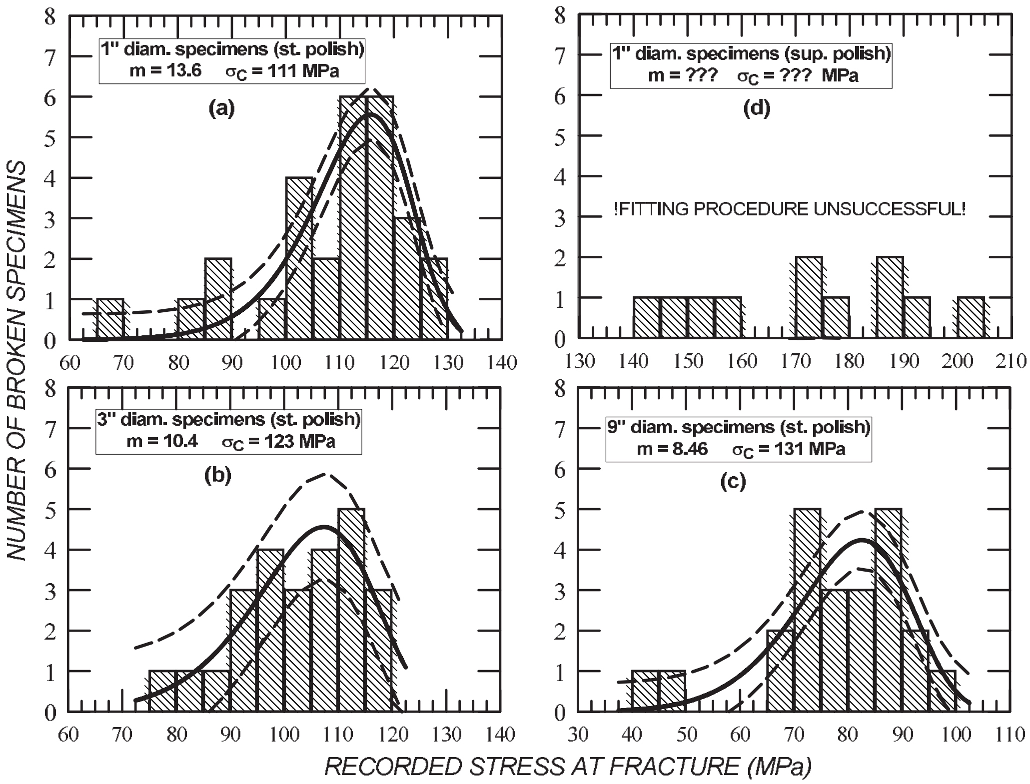

Fig. 3 Analysis of the failure probability density of the four sets of fused SiO_2 specimens fracture-tested at UDRI. The broken lines are indicative of the 95% confidence bands based on fitting the histograms to Eq. (12). The procedure failed with superpolished specimens due to insufficient test data.

Table 1) are displayed in Fig. 4(a), plotted against the biaxially stressed area. There is no evidence of any significant dependence on the stressed area or the surface polish, which leads us to conclude that, for practical purposes, it should be acceptable to write $m \simeq 10$, considering that an arithmetic average indicates

$$m = 9.89 \pm 1.42. \tag{13}$$

Keeping in mind that the parameter m characterizes the scatter—or spread—of recorded failure stresses, it follows that the flaw-size dispersion should be fairly homogeneous, i.e., essentially independent of the stressed area or the surface finish; the dispersions highlighted in Fig. 3 (~ 50 MPa) substantiate this point.

4.2 *Characteristic Strength*

In principle, we expect as-received specimens of fused SiO_2 subjected to tensile stresses acting on areas of different dimensions to exhibit identical characteristic strengths. Figure 4(b) emphasizes this is not the case here. For this reason, since the strength of an optical glass strongly depends on the surface finish, especially the level of subsurface damage (see, for instance, Ref. 13) we will take advantage of PBS measurements[15] to assess the surface quality of the specimens investigated at UDRI. Table 2 summarizes the results of examining available data, including relevant failure stresses (see the Appendix). First, note that the measured strengths (average of the recorded failure

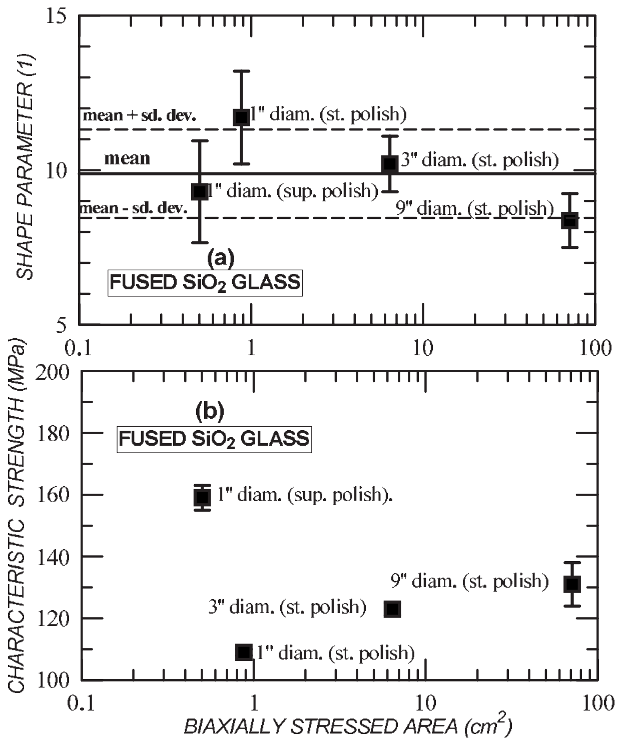

Fig. 4 Weibull statistical parameters of fused SiO_2 glass plotted against the equibiaxially stressed area: (a) shape parameter and (b) characteristic strength.

Table 2 Surface-finish characterization of fused SiO_2 specimens fracture-tested at UDRI.

	1-in. diam.	3-in. diam.	9-in. diam.	1-in. Repolished
Number of evaluated specimens (1)	21	17	8	5
Average PBS number[a] (ppm/sr)	17.8±3.7	11.8±3.6	11.7±3.5	0.199±0.095
Measured flexural strength[a] (MPa)	111±10	102±11	76.1±20.6	183±16

[a]average±standard deviation of the numbers listed in Table 3.

stresses for PBS-examined specimens) agree with measured strengths, as listed in Table 1, thus confirming that the samples are representative of the fracture-tested lots. The averages of mean PBS numbers, therefore, strongly suggest that the surface finish of 1-in.-diam specimens does not match that of 3- or 9-in.-diam specimens.

In this light, we plot (see Fig. 5) the characteristic strength against the average mean PBS number, including the result for repolished material and note a striking correlation. In effect, the correlation is linear:

$$\sigma_C(\text{in megapascals}) \simeq 160 - 2.83 \times \overline{\text{PBS}}(\text{in parts per million per steradian}), \tag{14}$$

thus providing an appropriate formulation of the dependence of inherent strength on surface quality. We see that the inherent strength (1-cm^2 stressed area) of superpolished fused SiO_2 (~ 160 MPa) substantially exceeds the strength of standard polished material (~ 110 MPa), which confirms the effectiveness of superpolishing, in accord with observations previously reported for sapphire.[16]

5 Failure Probability

The prime purpose of extracting Weibull parameter values from flexural strength data is to obtain means of assessing the failure probability of brittle components subjected to tensile stresses. Based on UDRI-generated data, we take it that setting the shape parameter equal to 10, independently of stressed area or surface-finish considerations, should be acceptable in dealing with windows or mirrors made of fused SiO_2 glass. Regarding the characteristic strength and its dependence on surface finish as formulated in Eq. (14), keep in mind that there is no correlation between the failure stress and the mean PBS number pertaining to a single specimen (see Table 3), which implies that, unless PBS measurements were performed on a suitable sample—say, eight or more identical specimens—the appropriate way to proceed is to take advantage of Eq. (14) with

$$\overline{\text{PBS}} = \begin{cases} 0 & \text{superpolish} \\ 15 & \text{standard polish,} \end{cases} \tag{15}$$

in accord with the plot displayed in Fig. 5.

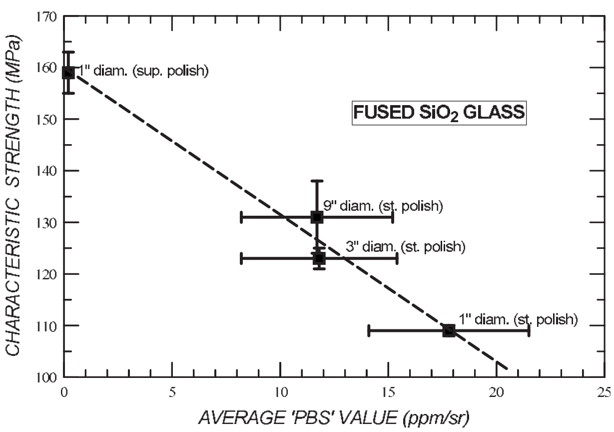

Fig. 5 Characteristic strength of fused SiO_2 glass as a function of the average mean PBS number for each of the four lots investigated at UDRI. The mean PBS number measures the degree of surface/subsurface imperfection of a given specimen. The broken line illustrates a linear fit, thus emphasizing the dependence of σ_C on $\overline{\text{PBS}}$.

Returning now to Eq. (11) for the cumulative failure probability, the case of an SiO_2 glass window subjected to uniform equibiaxial stresses over an area S is best illustrated as in Fig. 6. On setting $\overline{\text{PBS}}$ equal to 15, the critical stresses at the 1% failure probability level range from about 50 to 80 MPa, depending on the stressed area (0.3 $cm^2 \leq S \leq 100$ cm^2); with the superpolished material, the relevant numbers are 70 and 110 MPa. Evidently, relying on a "handbook-style" modulus of rupture in the context of designing SiO_2 glass components can be very misleading.

Similarly, Eq. (12) yields the failure probablity density distributions displayed in Fig. 7. As the stressed area increases from 0.3 to 100 cm^2 the peak position of the distribution shifts from about 130 to 70 MPa, thus reflecting the enhanced probability of capturing large fracture-initiating flaws; as expected, with the superpolished material the peak positions are seen to move substantially to the right. The spread of the distributions, however, decreases as the stressed area increases, which results in reduced skewness, as evidenced in Fig. 3 for 3- and 9-in.-diam specimens tested at UDRI.

6 Conclusion

The recent availability of a rich set of fracture-stress measurements performed on amorphous SiO_2 glass created an opportunity to better characterize the strength features of this material. These measurements were carried out in a ring-on-ring testing configuration and, therefore, are amenable to a description based on a Weibull model that rests on the concept of a characteristic strength (1-cm^2 uniformly stressed area). The procedure provided the means of clearly identifying the impact of the stressed area, to assess the

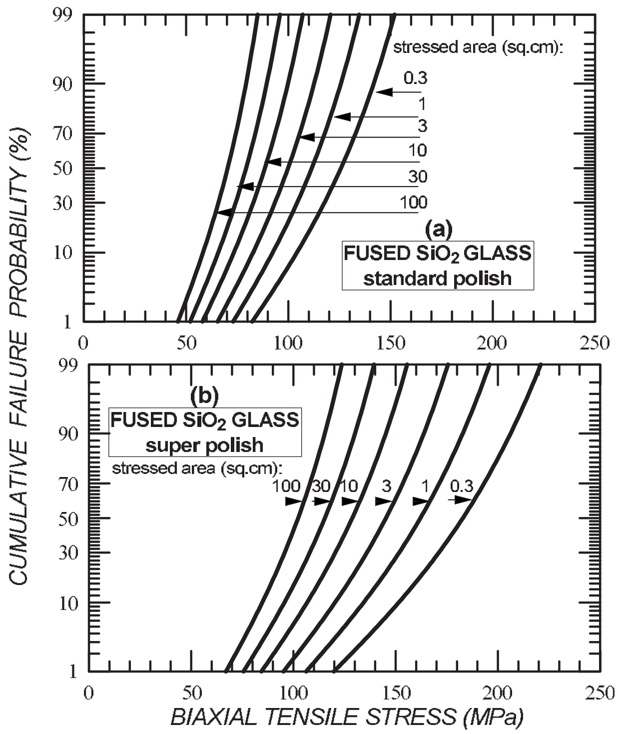

Fig. 6 Cumulative failure probability of fused SiO_2 windows subjected to equibiaxial uniform tensile stresses over surface areas ranging from 0.3 to 100 cm^2. The two plots are based on Eq. (11) with m set equal to 10. (a) Standard polish (σ_C=110 MPa) and (b) super polish (σ_C=160 MPa).

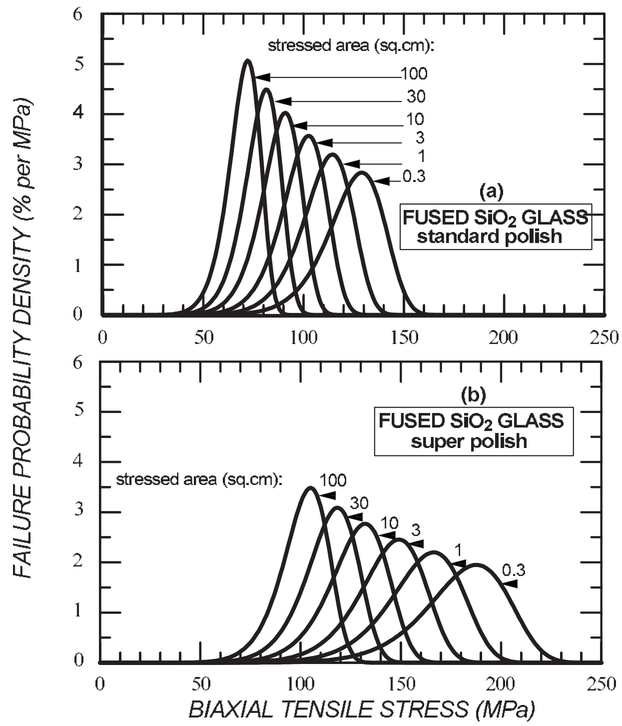

Fig. 7 Failure probability density of fused SiO_2 windows subjected to equibiaxial uniform tensile stresses over surface areas ranging from 0.3 to 100 cm^2. The two plots are based on Eq. (12) with m set equal to 10. (a) Standard polish (σ_C=110 MPa) and (b) super polish (σ_C=160 MPa).

effect of surface polishing, and hence to describe essential features of the fracture-strength behavior of fused silica or fused quartz.

A Weibull statistical analysis of failure-stress data—as was performed here—amounts to obtaining the parameters σ_C (characteristic strength) and m (shape parameter), which is best done by directly fitting estimated cumulative failure probability numbers to a failure probability expression, as formulated in Eq. (11). This procedure avoids distorting the distribution through logarithmic linearization and leads to the conclusion that, in effect, the Weibull shape parameter of fused SiO_2 is essentially independent of both the stressed area and the surface finish ($m \simeq 10$). In terms of flaw-size distributions, this implies that the spread of critical flaw sizes (not the sizes per se!)[‡] should be fairly homogeneous, in accord with the failure-stress distributions displayed in Fig. 3.

As expected, the characteristic strength of fused SiO_2 strongly depends on the surface finish. In this regard, the availability of PBS measurements to specify the level of surface/subsurface damage represents a major step forward in the sense that it leads to Eq. (14) to describe the inherent strength of fused SiO_2 specimens. In conjunction with a shape parameter set to 10 and given the appropriate $\overline{\text{PBS}}$ number, this equation can be taken advantage of to predict the cumulative failure probability as well as the failure probability density of a fused silica or fused quartz component subjected to uniform equibiaxial tensile stresses over a known area.

[‡]For an estimate of the flaw dimensions, see the Appendix in Ref. 17.

Appendix

PBS measurements provide an outstanding diagnostic tool for evaluating the finish of transparent polished surfaces. Note, however, that the mean PBS number of a given test specimen—as obtained over the gauge area—does not directly relate to the recorded stress at fracture. This is highlighted in Table 3, which lists the totality of available mean PBS numbers in conjunction with each specimen's fracture "strength." Nevertheless, the average mean PBS number of each lot correlates remarkably well (see Fig. 5) with the characteristic strength, as extracted from the failure probability, thus demonstrating the inherent stochastic nature of the fracture process. In this connection, note that Student's t testing we performed on the mean PBS numbers listed in Table 3 strongly suggests that the numbers for 3- and 9-in.-diam specimens belong to the same distribution (prob.=0.9689), whereas the 1-in. and 3- or 9-in. distributions are not related (prob.=0.0005). In summary, available evidence points to the conclusion that—on average—the 1-in.-diam SiO_2 glass specimens investigated at UDRI exhibit substantially more stress-concentrating imperfections than the 3- or 9-in.-diam specimens, thus leading to a straigthforward explanation of the characteristic strength "discrepancy."

Table 3 Mean PBS number (in parts per million per steradian) of each fused SiO_2 specimen examined at UDRI in conjunction with the recorded stress at fracture listed in ascending order.

1-in. diam.		3-in. diam.		9-in. diam.		1-in. Repolished	
PBS #	σ_i (MPa)	PBS #	σ_i (MPa)	PBS #	σ_i (MPa)	PBS #	σ_i (MPa)
17.1	86	16.7	75	10.7	43	0.329	158
14.7	98	13.5	85	7.5	47	0.170	179
14.4	100	7.5	91	17.0	73	0.2649	189
17.5	103	14.5	95	8.7	82	0.115	190
12.8	104	10.7	96	14.1	85	0.117	201
25.1	105	10.3	97	13.9	89		
10.9	109	11.8	98	14.1	91		
16.2	112	11.7	104	7.8	99		
13.2	113	5.9	104				
19.3	113	8.2	107				
19.4	113	8.0	108				
20.5	114	20.9	109				
17.8	115	12.3	110				
25.4	116	12.3	112				
19.2	116	11.9	114				
22.2	116	13.0	114				
15.4	122	11.0	116				
16.7	122						
16.4	122						
20.4	125						
18.5	125						

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