STRESSES IN A PIPE BEND OF OVAL CROSS-SECTION AND VARYING WALL THICKNESS LOADED BY INTERNAL PRESSURE

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ABSTRACT

The types of geometrical irregularity arising from the production of pipe bends are briefly discussed and formulae are presented which facilitate the calculation of stresses caused by each individual irregularity, when the pipe is subjected to internal pressure.

The individual formulae are combined to enable the stress distribution to be calculated in a pipe bend under internal pressure with all the irregularities discussed. Results given from the formulae for a typical pipe bend are compared with results obtained by the finite element method.

The requirements and limitations of British Standards are discussed in comparison with the predictions of the formulae derived in this paper.

NOMENCLATURE

a	Semi-major axis of ellipse, internal dimension.
b	Semi-minor axis of ellipse, internal dimension.
D	Mean internal diameter.
E	Young's modulus.
(E)	Elliptic integral of second kind.
e	Eccentricity.
(K)	Elliptic integral of first kind.
k	$\sqrt{\dfrac{a^2 - b^2}{a^2}}$
M_0	Fixing moment.

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p	Internal pressure.
r_i	Inside radius.
r_o	Outside radius.
r_m	Mean radius.
s	Arc length.
t	Thickness.
t_i	Wall thickness at intrados of bend.
t_o	Wall thickness at extrados of bend.
t_m	Mean wall thickness.
t_θ	Wall thickness at any angle, θ.
U	Strain energy in bending.
u_x, u_y	Displacements.
X	Deviation from radius of mean cross-section.
x, y	Co-ordinate directions.
ε_x, ε_y	Strains.
ξ, η	Bipolar co-ordinates.
ψ, ϕ	Angles.
θ	Angle around the cross-section measured outwards from major axis.
ν	Poisson's ratio.
σ_θ	Hoop stress.
σ_m	Membrane stress.
σ_b	Bending stress.

Any other notation is defined as it appears.

1. INTRODUCTION

During the manufacture of pipe bends the cross-section of the pipe may flatten considerably in the plane of the bend due to the bending operation. This flattened cross-section may take a variety of forms depending on the tools used to perform the bending operation, but often the resulting shape approximates to an ellipse. Also during bending, plastic flow in the wall of the pipe causes wall thinning on the outside of the bend (extrados) and thickening on the inside (intrados). These effects are shown in Fig. 1.

When the pipe bend is subjected to internal pressure these effects give rise to variations in stress around the cross-section, the overall result of which is to produce maximum stresses higher than those which one would expect to find had these effects not been present. In addition to this, the very nature of a pipe bend, as a section of a toroid, induces stresses which would not be present if the pipe were straight.

To obtain a complete theoretical solution to this problem is a very difficult task and an alternative method is to obtain a solution for each geometrical irregularity

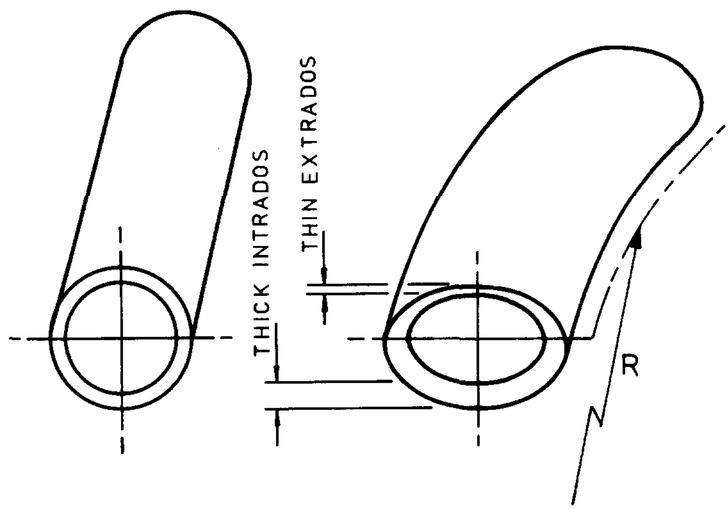

Fig. 1. Features of pipe before and after bending.

when considered independently and then, with caution, superpose these independent results to obtain an approximate solution to the problem. This paper commences by deriving formulae for the hoop stresses in:

(i) a straight tube of elliptical cross-section
(ii) a straight tube of circular cross-section but with variations in wall thickness
(iii) a circular section toroid

The solutions obtained from these formulae are compared with the alternative solutions available, including the results of a finite element analysis. Finally, the formulae for each individual irregularity are combined to enable stresses to be calculated in a pipe bend having all the geometrical irregularities mentioned.

Conclusions are drawn about the accuracy and usefulness of the solutions in comparison with the results which would be obtained by the use of British Standards, where the effects of ovality are not considered quantitatively.

2. OVALITY OF CROSS-SECTION

A solution to this problem was suggested by Haigh[1] in 1936.

Suppose that the mean radius r_1 of an oval cylinder can be described by the nominal radius plus a cosine term (see Fig. 2), i.e.

$$r_1 = \frac{D}{2} + X_1 \cos 2\theta$$

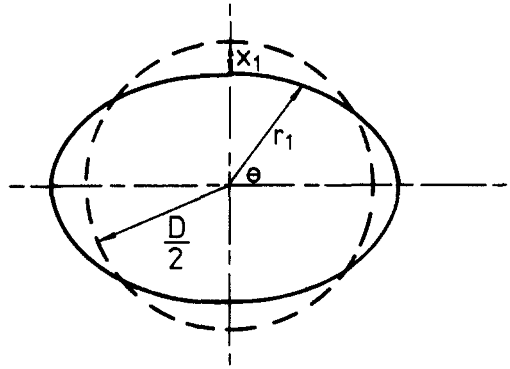

Fig. 2. Notation for Haigh's analysis.

When subjected to internal pressure the cylinder will tend to become more circular in shape, giving a new value for the radius, assumed by Haigh to be:

$$r_2 = \frac{D}{2} + X_2 \cos 2\theta$$

By considering the equilibrium of an element of the cylinder wall, the change in ovality after pressurisation, $X_1 - X_2$, can be found in terms of the initial ovality, X_1, only. The change in curvature evaluated from this can be used to find the bending stress in the cylinder wall by substitution into the simple bending equations.

An expression for the bending stress in the cylinder wall obtained by this method is given by:

$$\sigma_b = \frac{3X_1pD}{t^2} \cdot \left[\frac{1}{1 + \frac{p(1 - \nu^2)}{E}\left(\frac{D}{t}\right)^3} \right] . \cos 2\theta$$

(A full derivation of this formulae is given in reference 2, page 39). In order to check the results obtained from Haigh's method it seemed desirable to seek some alternative method of solution and an obvious choice was to apply Castigliano's theorem using the strain energy of bending.

The accuracy of the results obtained by applying Castigliano's theorem to rings and cylinders is well documented and so its application in this particular case provides a useful yardstick against which results from Haigh's method and from finite element calculations can be compared.

This analysis was not found in any other literature, and for this reason it is given here in full.

Consider the bending moment at section xx as shown in Fig. 3.

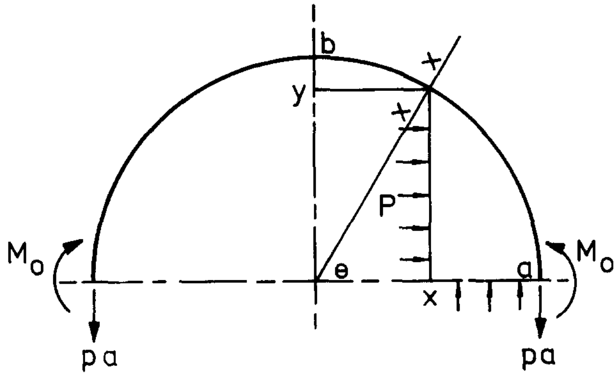

Fig. 3. Notation for strain energy analysis.

$$M_{xx} = M_0 - pa.(a-x) + p(a-x).\frac{(a-x)}{2} + py.\frac{y}{2} \tag{1}$$

from the equation for an ellipse.

$$r^2 = \frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

from Fig. 3

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{2}$$

from eqn. (1):

$$\frac{\partial M_{xx}}{\partial M_0} = 1$$

by Castigliano's theorem

$$\frac{\partial U}{\partial M} = \phi$$

where:

$$U = \int \frac{M^2}{2EI}\,\mathrm{d}s$$

At the point $\theta = 0$, $M_{xx} = M_0$ and $\phi = 0$.

$$\phi = 0 = \frac{1}{EI}\int_0^{\pi} M_{xx}\frac{\partial M_{xx}}{\partial M_0}\,\mathrm{d}s$$

Now $\mathrm{d}s = r.\mathrm{d}\theta$

$$\phi = 0 = \frac{1}{EI}\int_0^\pi \left(M_0 - pa\,.\,(a-x) + \frac{p}{2}\,.\,(a-x)^2 + \frac{py^2}{2}\right) r\,.\,\mathrm{d}\theta$$

from eqn. (2):

$$0 = \int_0^\pi \left(M_0 r - pa^2 r + par^2\cos\theta + \frac{p}{2}a^2 r + \frac{pr^3}{2}\cos^2\theta - \frac{p}{2}\,.\,2ar^2\cos\theta + \frac{pr^3}{2}\sin^2\theta\right).\,\mathrm{d}\theta = \int_0^\pi M_0 r - \frac{p}{2}a^2 r + \frac{p}{2}r^3\,.\,\mathrm{d}\theta$$

Now remembering that $r = f(\theta)$:

$$0 = \int_0^\pi M_0\left(\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\right)^{1/2} - \frac{p}{2}a^2\left(\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\right)^{1/2} + \frac{p}{2}\left(\frac{a^2b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\right)^{3/2}.\,\mathrm{d}\theta$$

It is now possible to find the value of M_0, the unknown fixing moment, by evaluating the three terms of the integral independently. This is done by transforming the first two terms and the last term into complete elliptic integrals of the first and second kinds, respectively.

This finally gives integrals of the form:

$$0 = 2M_0 b\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\psi)^{1/2}}\mathrm{d}\psi - pa^2 b\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\psi)^{1/2}}\mathrm{d}\psi + pb^3\int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\psi)^{3/2}}\mathrm{d}\psi$$

where:

$$\psi = \frac{\pi}{2} - \theta \quad \text{and} \quad k^2 = \frac{a^2 - b^2}{a^2}$$

After inserting the values of k^2 appropriate to the problem the values of the definite integrals may be obtained from tables of complete elliptic integrals under the headings (K) and (E) for the first and second types, respectively.

The final form of the integrals is given as:

$$0 = 2M_0 b(K) - pa^2 b(K) + \frac{pb^3}{1-k^2}(E)$$

and the unknown fixing moment is found to be of the form:

$$M_0 = (\text{constant})\,.\,(pa^2)$$

Hence, by substituting the value of M_0 into eqn. (1) the value of the bending moment, M_{xx}, and consequently the stress, may be found for any point on the elliptical wall.

3. VARIATION OF WALL THICKNESS

A tube with the same variation of wall thickness as that found in a pipe bend can be considered as a cylinder with an eccentric bore. Bending stresses are induced in the wall of the cylinder when it is subjected to internal pressure but the calculation of these stresses is a difficult task and has only been carried out successfully by using the methods of Jefferey[3] utilising the bipolar co-ordinates and real stress functions or those of Timoshenko and Goodier[4] using complex potential stress functions. The results obtained by either method are identical.

The maximum stresses in the cylinder wall are the hoop stresses on the inside face. Using Timoshenko and Goodier's notation, these are given by:

$$\sigma_\theta = -p + 2p\frac{(\cosh\xi_i - \cos\eta)}{(\sinh^2\xi_i + \sinh^2\xi_0)}\,.(\sinh\xi_i\coth(\xi_i - \xi_o) + \cos\eta)$$

where any point on the cross-section of the cylinder is defined in terms of the bipolar co-ordinates ξ and η where:

$$x + iy = ia\coth\zeta/2$$

and

$$\zeta = \xi + i\eta$$

The subscripts i and o refer to the co-ordinates of the inside and outside boundaries, respectively.

(A full analysis of complex potential stress functions and their application to this specific problem is given in reference 2, Appendix 4.A.1.)

Calculations using Timoshenko and Goodier's approach show that for wall thinning ratios of:

$$t_{\max}/t_{\min} \leq 3$$

the bending stresses are small (less than 6 % of the membrane stresses). If the bending stresses are neglected, then the distribution of stress around the cylinder wall may be calculated from:

$$\sigma_\theta = \frac{pd}{2t}$$

where t is the actual thickness at any angle θ, as shown in Fig. 4, and may be found from:

$$t^2 + t(2r_i + 2e\sin\theta) + (r_i^2 - r_o^2 + e^2 + 2er_i\sin\theta) = 0$$

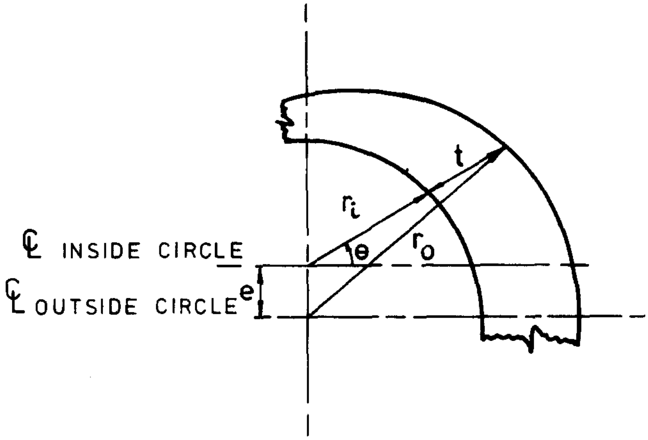

Fig. 4. Part of an eccentric bore cylinder.

Alternatively, Lamé's method may be used to calculate the hoop stresses, by assuming, for instance, that t and r_i are correct and the effective $r_o = r_i + t$. The justification of this latter method lies in comparisons with finite element results, not given in detail in this paper.

4. EFFECTS DUE TO TOROIDAL SHAPE

The bending stresses present in the wall of a circular section toroid may be evaluated either by solving exactly the governing equations for an element of the cylinder wall or by applying the theory of axisymmetric thin shells in which some of the governing equations are simplified. However, the problems which are to be encountered in formulating and solving the differential equations in this kind of approach are of such proportions that a membrane solution to this problem is usually regarded as acceptable, although thin shell solutions have been obtained by some workers in the field (see for example reference 5).

A membrane solution is obtained by considering the vertical equilibrium of an element of the toroid wall (see Fig. 5).

It can be shown that:

$$\pi[(R + a)^2 - R^2].p = 2\pi(R + a)t\sigma_\theta \sin\theta$$

Simplifying:

$$\pi pa(a + 2R) = 2\pi t\sigma_\theta(R + a)\sin\theta$$

$$\sigma_\theta = \frac{pa(a + 2R)}{2t\sin\theta(a + R)}$$

but $a = r\sin\theta$, giving:

$$\sigma_\theta = \frac{pr}{2t} \cdot \left[\frac{r\sin\theta + 2R}{r\sin\theta + R}\right]$$

This result is simply a product of the thin shell membrane stress for a straight tube and a factor which may be greater or less than one depending on the sign of θ.

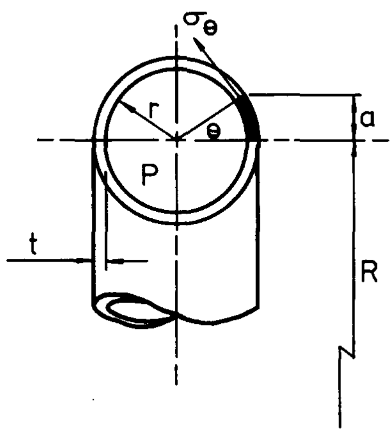

Fig. 5. Section of a toroidal shell.

It should be noted that the membrane solution leads to a displacement discontinuity at the ends of the diameter which is perpendicular to the plane of bending. Since this discontinuity cannot occur, local bending stresses are created in this region but, apart from the finite element method, the values are not easily found. This effect is illustrated clearly in Fig. 9.

5. COMPARISON OF RESULTS

A particular pipe bend was chosen, typical of those found in high pressure steam generating plant, so that comparisons could be made between the various analytical solutions and a finite element solution. The dimensions and properties of the bend are as shown in Fig. 6.

The three types of geometrical irregularity were then considered individually and the appropriate methods of solution applied to each, using the following data.

(i) A straight tube of elliptical cross-section and uniform wall thickness:

$$a = 68{\cdot}5\,\text{mm}$$
$$b = 60{\cdot}9\,\text{mm}$$
$$t = t_m = 16\,\text{mm}$$

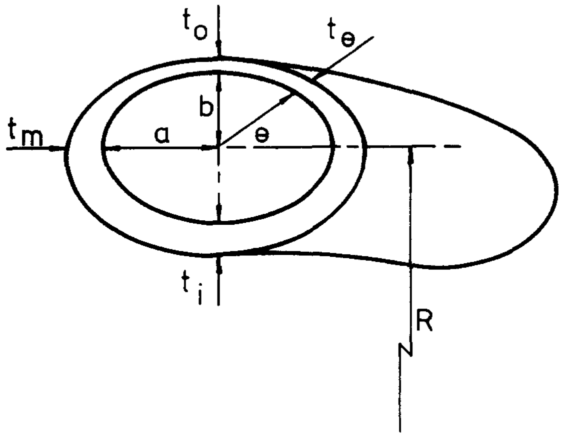

Fig. 6. Notation for analysis of a complete pipe bend.

(ii) A straight circular tube with varying wall thickness:

$$a = b = 68{\cdot}5\,\text{mm}$$
$$t_i = 24\,\text{mm}$$
$$t_o = 8\,\text{mm}$$

The grossly exaggerated wall thinning was found necessary in order to make a good comparison between the two analytical methods.

(iii) A circular section toroid with uniform wall thickness:

$$a = b = 68{\cdot}5\,\text{mm}$$
$$t = t_m = 16\,\text{mm}$$
$$R = 457\,\text{mm}$$

In each case, the respective graphs (Figs. 7, 8 and 9) show the distribution of hoop stress around the circumference of the section. In most cases satisfactory agreement is obtained between the various alternative solutions, particularly in the regions where the greatest stresses are to be found. The only exception to this is in the case of the thin shell toroid, where the analytical solution clearly neglects any bending stresses which may be present. A closer approximation to the true maximum stress may be obtained if the membrane stress is calculated using the mean, instead of the internal, dimensions. In the case of cross-section ovality, the solutions obtained by Haigh's method and by strain energy methods show a very close agreement, but for most practical purposes the solution due to Haigh would be considered preferable as it requires less analysis and does not require the use of the rather specialised tables of elliptic integrals.

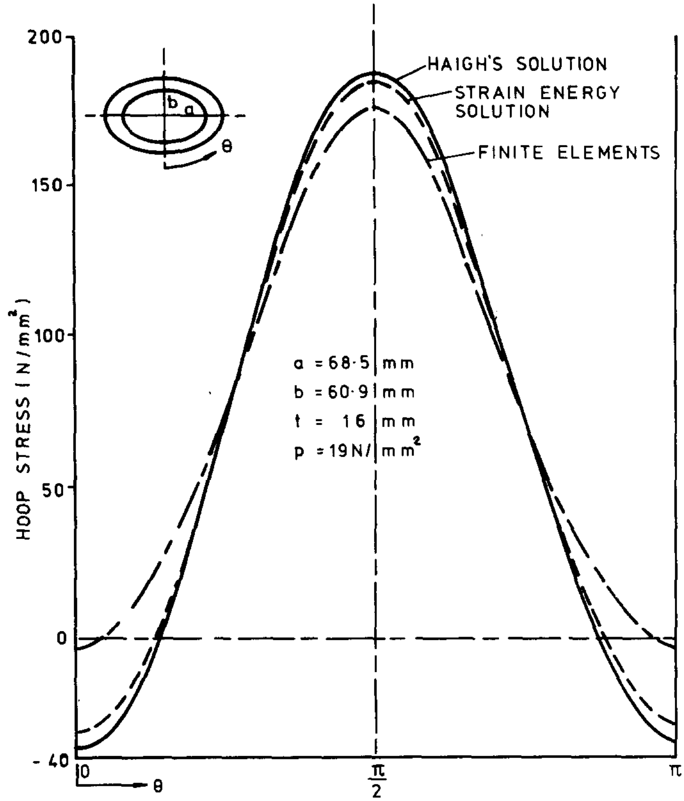

Fig. 7. Stresses in a cylinder of elliptical cross-section (inside face).

Even with the greatly exaggerated wall thinning ratios of 3 to 1 the membrane solution compares well with the exact stress function solution and it can be seen from Fig. 8 that the bending stresses account for only a small proportion of the total stresses in the cylinder wall.

6. THE EFFECT OF HIGHER ORDER STRAIN COMPONENTS

All the methods considered in the other sections of this paper are based on the linear theory of elasticity in which the strains and displacements are assumed small. The inference in the latter case is that the dimensions of the body do not significantly change. This may not be the case for rod-like or plate-like structures, where, although the strains are small, the displacements may be large enough to modify the linear theory, if membrane forces are present. The classical theory of compression

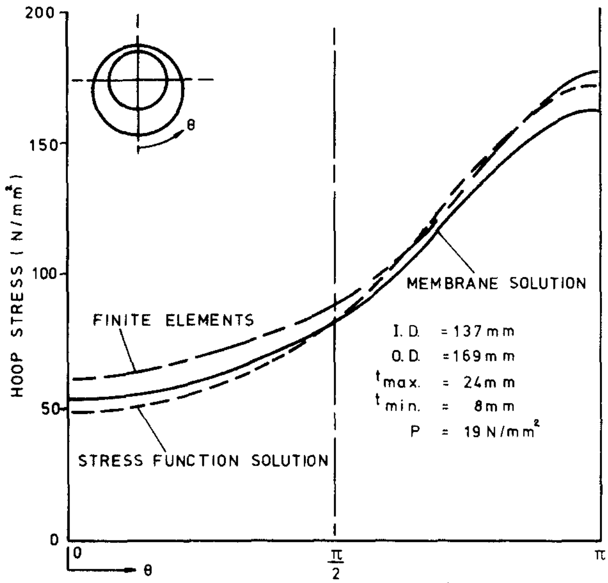

Fig. 8. Stress on the inside face of an eccentric bore cylinder.

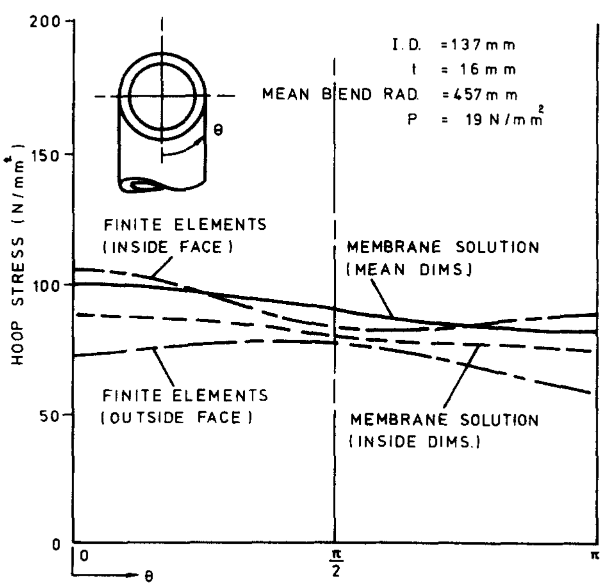

Fig. 9. Stresses in a circular toroid.

instability (e.g. the well known Euler and eccentric struts) takes account of the lateral deflections of the rod or plate, which may be sufficient to cause appreciable increases in the bending moments due to the external forces. In the case of the wall of an oval pipe the effect is reversed and the tensile membrane stresses will tend to reduce the lateral deflections due to internal pressure.

There appears to be no analytical solution for the case of the oval pipe, nor indeed is there any reliable method of estimating if the effect of the membrane stresses is appreciable. A possible approach is to use the finite element method modified to allow for the change in geometry of the cross-section. The standard small deflection method has to be modified in two respects; first by increasing the pressure in steps and modifying the section co-ordinates for each step and, secondly, by introducing the higher order strain components. In the present case this means replacing the expressions:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} \quad \text{and} \quad \varepsilon_y = \frac{\partial u_y}{\partial y}$$

by

$$\varepsilon_x = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2 \quad \text{and} \quad \varepsilon_y = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\frac{\partial u_x}{\partial y}\right)^2$$

This change allows for the rotations of the elements to be appreciable, although the strains are still small. The other higher order strain components are still negligible unless the strains themselves become large, which is unlikely in the present application. A simple linear solution is precluded and an iterative approach is necessary. It is obviously useful to be able to estimate the accuracy of a linear solution and a common method of assessment is to compare the lateral displacements with the thickness of the wall. In the example chosen in Section 5, the maximum lateral displacement is of the order of 1 % of the pipe wall thickness. It is probable that in this case the effects of change of geometry and the higher order strain components are negligible and that the linear elastic solution is accurate. This might not be the case, however, if large deformations due to creep were being considered.

7. UNIFICATION OF FORMULAE

When the three individual effects are present together, some method of combining the individual formulae has to be used. The technique usually adopted in thin shell analysis is to obtain two individual solutions, one for membrane stresses only and the other for bending stresses only. A complete solution is then obtained by simply adding the two together. Adopting this technique, the only bending stresses considered are those due to ovality of cross-section and so Haigh's result alone provides half of the complete solution. The membrane stresses may be obtained by using the formula for a thin shell toroid given in Section 4 but the wall thickness used in the calculation will be that given by the formula in Section 3, where an eccentric bore tube was considered.

The formulae needed for a complete solution are given below and a pictorial definition of the symbols is given in Fig. 6.

Let

$$r_i = \frac{a+b}{2} \qquad r_m = r_i + \frac{t_m}{2}$$

and t_θ = thickness for a given angle, θ. Then the bending stress due to Haigh is given by:

$$\sigma_b = \frac{3pr_i(a-b)}{t_m^2} \cdot \left[\frac{1}{1 + \dfrac{(1-\nu^2)}{2E}\left(\dfrac{2r_i}{t_m}\right)^3} \right] \cos 2\theta$$

while the toroid membrane stress is given by:

$$\sigma_m = \frac{pr_m}{2t_\theta} \cdot \left[\frac{2R + r_m \sin\theta}{R + r_m \sin\theta} \right]$$

The addition of these gives a total hoop stress:

$$\sigma_\theta = p \left\{ \frac{r_m}{2t_\theta} \cdot \left[\frac{2R + r_m \sin\theta}{R + r_m \sin\theta} \right] + \frac{3r_i(a-b)}{t_m^2} \left[\frac{1}{1 + \dfrac{p(1-\nu^2)}{2E}\left(\dfrac{2r_i}{t_m}\right)^3} \right] \cos 2\theta \right\}$$

Figure 10 shows a comparison of the results obtained by this method and by a finite element analysis for the pipe bend with the following dimensions:

$a = 68{\cdot}5$ mm
$b = 60{\cdot}9$ mm
$R = 457$ mm
$t_i = 17{\cdot}8$ mm
$t_o = 14{\cdot}2$ mm
$t_m = 16$ mm

when subjected to an internal pressure of 19 N/mm^2.

It has been assumed throughout this analysis that the maximum hoop stresses will occur on the inside face of the tube and in general this is so. If, however, doubts do exist about the location of the greatest stresses it is a fairly simple task to re-arrange the formulae to give stresses on the outside face of the tube.

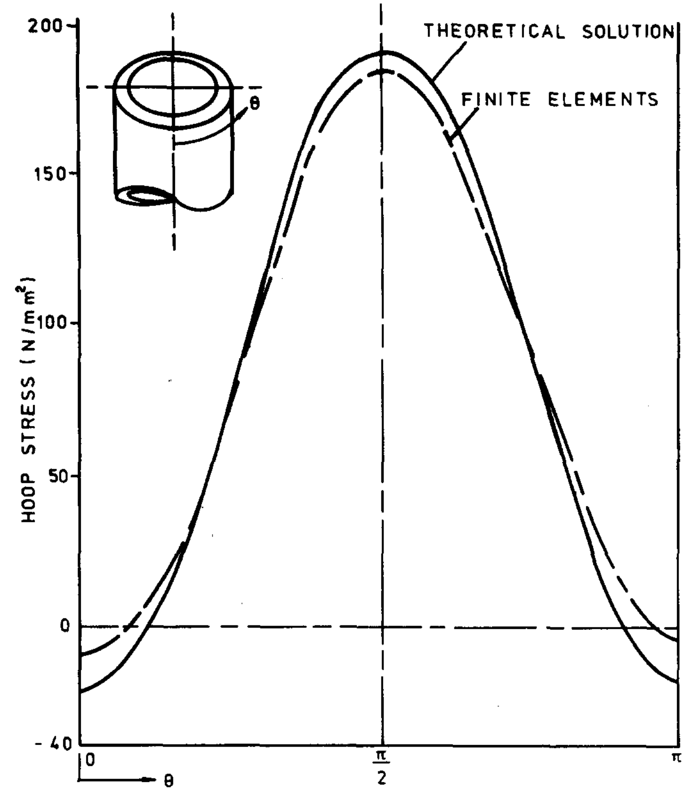

Fig. 10. Stresses in a complete pipe bend (inside face).

8. DISCUSSION

Although all three geometrical irregularities contribute to the variation of stress in pipe bends it can be seen by comparing Figs. 7 and 10 that the overall stress distribution is most greatly influenced by the ovality of the pipe cross-section. Because of this, care must be exercised when calculating the dimensions of a pipe required for producing a safe bend, particularly if the Standards being used (e.g. BS806) appear to make no quantitative allowance for the effects of ovality.

For example, consider a system of pipework made up of straight pipes with dimensions similar to those of the pipe bend considered in Section 7 and designed to work at a maximum stress level of $100\,N/mm^2$. BS806 requires that the wall thickness of any pipe used to manufacture a bend for the same pipework system be greater than that of a straight pipe by a factor of 1·125. This increase in wall thickness satisfactorily allows for the toroidal and wall thinning effects in the pipe bend and an analysis based on these two irregularities alone suggests a stress of not greater than 96 N/mm^2, i.e. within the 100 N/mm^2 design limit.

Section 9.3 of BS806 requires that the ovality of cross-section of a pipe bend shall not exceed 5% but gives no indication of the stresses which will arise for any particular degree of deformation. In fact, when the allowable 5% ovality is included in the calculation, the new maximum stress in the pipe bend rises to over 136 N/mm^2, well outside the design limit. Although it has been shown that high stresses may occur in piping installation, it should be noted that failures attributable to the effects discussed in this paper do not happen as frequently as one would expect. One possible reason is that when stresses do occur which are large enough to be detrimental to the life of the pipe bend, rapid creep deformation may also take place, with the result that the cross-section becomes more circular and ovality stresses will be less significant.

An investigation into this problem is currently being undertaken.

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