2.2.4 Find expressions for the average kinetic, potential, and total energies of the vibration (2.2.4).

2.3 *Standing Waves on a Circular Membrane*

We have investigated, in Chap. 1, the normal-mode vibrations of a string segment with fixed ends and, in the preceding section, of a rectangular membrane with fixed boundaries. In each case, we based the analysis on a partial differential equation that described wave motion on the structure. We found that the method of variable separation leads to certain functions of position and of time, from which we are able to construct an infinity of normal-mode functions that satisfy the spatial boundary conditions of the problem, with sinusoidal time factors that can be adjusted to satisfy initial conditions. In particular, we found that the spatial boundary conditions inevitably lead to a discrete set of values (eigenvalues) of the separation constants, telling us the wave numbers and the frequencies of the normal-mode vibrations.

In the case of the two-dimensional rectangular membrane, we are able to satisfy the boundary condition of no displacement on all four edges of the rectangle because the edges coincide with the lines of constant x, or of constant y. That is, the method of variable separation, using cartesian coordinates, automatically gives functions that are suited to fitting boundary conditions along these coordinate lines.

To fit the boundary condition of no displacement on other than rectangular boundaries requires the use of an appropriate two-dimensional *orthogonal curvilinear coordinate system* such that the boundary of the membrane coincides with coordinate lines in this system. Furthermore, it is necessary that the variables of the wave equation be separable in the new system. It turns out that the choice of curvilinear coordinate systems is severely limited, and it is impossible, except in an approximate way, to analyze the vibrations of a membrane having an arbitrarily shaped boundary. A circular boundary, however, is a coordinate line of a polar coordinate system, and, as we shall see, it is possible to separate the variables of the wave equation in polar coordinates. The solution of one of the separated equations consists of Bessel functions; it is primarily to introduce these functions that we have chosen to investigate the vibrations on a circular membrane as a second example of two-dimensional normal-mode vibrations.

Our first task is to change the wave equation (2.1.1) from xy coordinates to polar coordinates r and θ, with the origin at the center of a circular membrane of radius a. According to Prob. 2.3.2, the wave equation then becomes

$$\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2} = \frac{1}{c_m{}^2}\frac{\partial^2 \zeta}{\partial t^2}. \tag{2.3.1}$$

We next assume that (2.3.1) has a solution of the form

$$\zeta(r,\theta,t) = R(r) \cdot \Theta(\theta) \cdot T(t) \tag{2.3.2}$$

and find, after multiplication through by $c_m^2/R\Theta T$, that

$$\frac{c_m^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{c_m^2}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = \frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2. \tag{2.3.3}$$

As before, we have introduced the separation constant $-\omega^2$, and again we find the differential equation (2.1.5) for the time function. The spatial part of (2.3.3) can now be rearranged to read

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \left(\frac{\omega}{c_m}\right)^2 r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = m^2, \tag{2.3.4}$$

where we have chosen to denote the second separation constant by m^2. Equation (2.3.4) thus separates into the two ordinary differential equations

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0 \tag{2.3.5}$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[\left(\frac{\omega}{c_m}\right)^2 - \frac{m^2}{r^2}\right] R = 0. \tag{2.3.6}$$

The equation for $\Theta(\theta)$ has the independent complex solutions $e^{\pm im\theta}$, or the independent real solutions $\cos m\theta$ and $\sin m\theta$. We see that the separation constant m must be either zero, which makes Θ a constant, or a (positive) integer, which makes Θ a single-valued function of θ. In effect, we are making use of a boundary condition along a radial line on the circular membrane, namely, that the displacement and its θ derivative be continuous functions across this hypothetical boundary. For the vibrations of a sector-shaped membrane, m could have other than integral values.

The differential equation for $R(r)$ can be put in the standard form

$$\frac{d^2R}{du^2} + \frac{1}{u}\frac{dR}{du} + \left(1 - \frac{m^2}{u^2}\right) R = 0 \tag{2.3.7}$$

by changing to the dimensionless independent variable

$$u = \kappa r = \frac{\omega}{c_m} r. \tag{2.3.8}$$

Equation (2.3.7) is known as *Bessel's equation*. Since it is of second order, it must have two linearly independent solutions for each value of the parameter m, which in the present instance we know to be a positive integer or zero. The two solutions of (2.3.7) are normally designated by $J_m(u)$ and $N_m(u)$. They are tabulated functions, just as $\cos\theta$ and $\sin\theta$ are two independent tabulated functions.†

The solution $J_m(u)$ is called the Bessel function (of the first kind) of order m, and it remains finite over the entire range of u from 0 to ∞. The other solution, $N_m(u)$, is called the Neumann function (or the Bessel function of the second kind) of order m, and it becomes infinite at $u = 0$ though it is finite elsewhere. Since $N_m(u)$ cannot represent a possible displacement of a circular membrane, we need only examine the properties of the functions $J_m(u)$. Neumann functions, however, are needed in discussing problems with other boundary conditions, such as the vibrations of an annular membrane.

The function J_m can be expressed by the infinite series

$$J_m(u) = \frac{u^m}{2^m m!}\left[1 - \frac{u^2}{1!2^2(m+1)} + \frac{u^4}{2!2^4(m+1)(m+2)} - \cdots\right], \tag{2.3.9}$$

found by assuming a series solution for (2.3.7) expanded about the origin. The numerical coefficient $1/2^m m!$ is purely conventional. A plot of $J_m(u)$ for $m = 0$, 1, 2 is given in Fig. 2.3.1. All the Bessel functions but J_0 vanish at the origin,

† For a brief account of Bessel functions, see M. L. Boas, "Mathematical Methods in the Physical Sciences," pp. 559–577, John Wiley & Sons, Inc., New York, 1966.

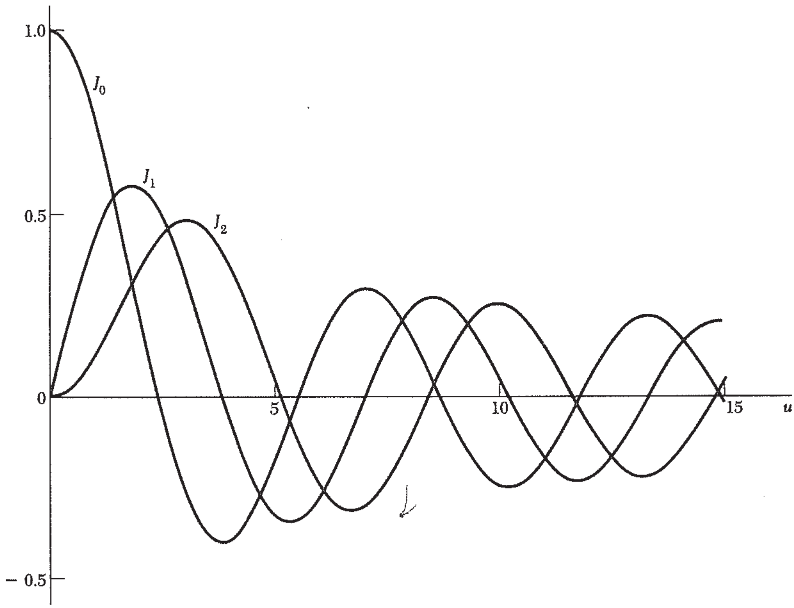

Fig. 2.3.1 Bessel functions of the first kind of order zero, one, and two.

TABLE 2.1 The nth Roots of $J_m(u) = 0$

$n \backslash m$	0	1	2	3
1	2.405	3.832	5.136	6.380
2	5.520	7.016	8.417	9.761
3	8.654	10.173	11.620	13.015
4	11.792	13.324	14.796	16.223

and $J_0(0) = 1$. Each Bessel function is seen to alternate in sign with increasing u, with its amplitude slowly dropping off (ultimately as $u^{-1/2}$), and with the spacing of its zeros becoming more nearly uniform (approaching π). The behavior reminds one of a damped sine wave. A few of the roots of $J_m(u) = 0$ are listed in Table 2.1. The roots of Bessel functions of adjacent orders interlace each other.

The Bessel functions obey *recursion relations*, such as

$$J_{m+1}(u) = \frac{2m}{u} J_m(u) - J_{m-1}(u) \tag{2.3.10}$$

$$\frac{dJ_m(u)}{du} = -\frac{m}{u} J_m(u) + J_{m-1}(u). \tag{2.3.11}$$

These relations may be established directly from the infinite series (2.3.9). They show that it is necessary to have numerical tables for only J_0 and J_1. The values of all higher-order Bessel functions, as well as all first derivatives, can then be calculated from the recursion relations.

Let us now see what the normal-mode vibrations of a circular membrane are like. When $m = 0$, Θ is independent of θ, so that

$$\zeta(r,t) = A\, J_0(\kappa r) \cos\omega t \tag{2.3.12}$$

is a possible solution of the wave equation. To satisfy the boundary condition that $\zeta(r,t) = 0$ at $r = a$, the value of $\kappa = \omega/c_m$ must be chosen to make

$$\kappa_{0n} a = u_{0n} \qquad n = 1, 2, 3, \ldots, \tag{2.3.13}$$

where u_{0n} is one of the roots of $J_0(u) = 0$, some of which are listed in the first column of Table 2.1. The frequencies of these radially symmetric $(0,n)$ modes are therefore

$$\omega_{0n} = \frac{c_m u_{0n}}{a}, \tag{2.3.14}$$

the lowest frequency being $\omega_{01} = 2.405(c_m/a)$.

When $m = 0$ and $n = 2$, there is a single nodal circle at the radius

$$r = \frac{u_{01}}{\kappa_{02}} = \frac{u_{01}}{u_{02}}\,a. \tag{2.3.15}$$

There are evidently $n - 1$ nodal circles when the nth root of $J_0(u) = 0$ coincides with the fixed boundary. They are at the radii

$$r = \frac{u_{0p}}{u_{0n}}\,a \qquad p = 1, 2, \ldots, n-1. \tag{2.3.16}$$

Next suppose that $m = 1$ and that we choose $\cos\theta$ for the Θ function. A solution of the wave equation is then

$$\zeta(r,\theta,t) = A\,J_1(\kappa r)\cos\theta\cos\omega t. \tag{2.3.17}$$

To satisfy the boundary condition at $r = a$ we must now have

$$\kappa_{1n}a = u_{1n} \qquad n = 1, 2, 3, \ldots, \tag{2.3.18}$$

where u_{1n} are the roots of $J_1(u) = 0$ appearing in the second column of Table 2.1. The frequencies of these normal modes are evidently

$$\omega_{1n} = \frac{c_m u_{1n}}{a}. \tag{2.3.19}$$

A nodal diameter exists at the angles $\theta = \pi/2, 3\pi/2$, as well as nodal circles at the radii

$$r = \frac{u_{1p}}{u_{1n}}\,a \qquad p = 1, 2, \ldots, n-1. \tag{2.3.20}$$

We could just as well have used $\sin\theta$ for $\Theta(\theta)$, or any linear combination of $\cos\theta$ and $\sin\theta$. That is, the nodal diameter can have any orientation, and its orientation depends on how the vibration is set up. We thus have a type of degeneracy which can be removed by stabilizing the orientation of the diameter by applying a *constraint* to the membrane at some point other than the center. The constraint forces the nodal diameter to pass through that point. We lose no generality by choosing $\cos\theta$ so long as we are free to pick the θ origin (polar axis) appropriately.

Our discussion of the various normal modes of a circular membrane can be readily extended to arbitrary values of m, with the outer boundary at $r = a$ such that $\kappa_{mn}a = u_{mn}$, where u_{mn} is the nth root of $J_m(u) = 0$. Evidently there are m nodal diameters, and $n - 1$ nodal circles. Figure 2.3.2 shows some of the possible modes of vibration of the circular membrane for small values of m and n.

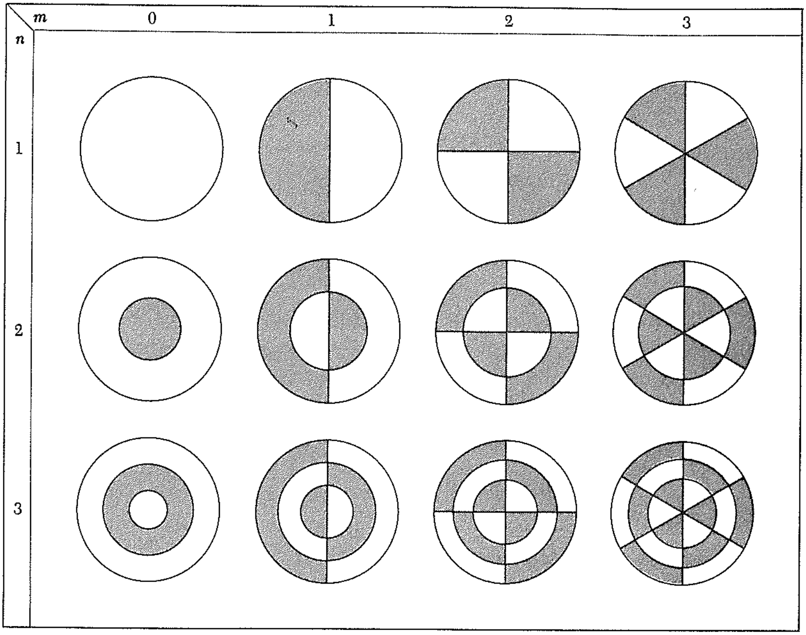

Fig. 2.3.2 Normal modes of the circular membrane.

Problems

2.3.1 Explain, in physical terms, why the method of separation of variables applied to the wave equations that we have considered always leads to a *sinusoidal* time function, though the spatial functions may take a variety of forms.

2.3.2 Use the relations $x = r\cos\theta$, $y = r\sin\theta$ connecting rectangular and polar coordinates to transform the wave equation in cartesian coordinates (2.1.1) to that in polar coordinates (2.3.1).

2.3.3 Assume a trial solution $R(u) = u^p \sum_0^\infty a_n u^n$ for the Bessel equation (2.3.7) and establish the series solution (2.3.9), except for the arbitrary numerical factor $1/2^m m!$.

2.3.4 Establish the recursion relations (2.3.10) and (2.3.11) from the series (2.3.9) or directly from the differential equation (2.3.7). Note the special case $dJ_0(u)/du = -J_1(u)$. Can you develop a recursion relation for the second derivative, $d^2J_m(u)/du^2$?

2.3.5 Show how to find the normal-mode frequencies of a membrane in the form of a semicircle with fixed boundaries along its edges.