INSTRUCTOR'S SOLUTIONS MANUAL

GAS DYNAMICS

THIRD EDITION

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Table of Contents

Preface

This manual contains the solutions to all 292 problems contained in *Gas Dynamics*, Third Edition.

As in the text example problems, spreadsheet computations have been used extensively. This tool enables more accurate, organized solutions and greatly speeds the solution process once the spreadsheet solver has been developed. To accomplish the solution of the text examples and problems in this manual nearly 40 separate spreadsheet programs were constructed. Some of these programs required only minutes to build, while others were more challenging.

The authors have attempted to carefully explain and detail the problem solutions so as to save time for the users. However, it should be recognized that some errors may have inadvertently crept into the manual. Should a user find any defects, the authors would appreciate hearing from the user so that revisions can be prepared. Please e-mail any comments to tkeith@eng.utoledo.edu

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Chapter One

BASIC EQUATIONS OF COMPRESSIBLE FLOW

Problem 1. – Air is stored in a pressurized tank at a pressure of 120 kPa (gage) and a temperature of 27°C. The tank volume is 1 m^3. Atmospheric pressure is 101 kPa and the local acceleration of gravity is 9.81 m/s^2. (a) Determine the density and weight of the air in the tank, and (b) determine the density and weight of the air if the tank was located on the Moon where the acceleration of gravity is one sixth that on the Earth.

$$P_{abs} = P_{gage} + P_{atm} = 120 + 101 = 221\,\text{kpa}$$
$$T = 27 + 273 = 300°C$$
$$\forall = 1\,m^3$$
$$g = 9.81\,m/s^2 \quad R = 0.287\,kJ/kg\cdot K$$

a) $$\rho = \frac{P}{RT} = \frac{221}{(0.287)(300)} = 2.5668\frac{kg}{m^3}$$

$$W = mg = \rho\forall g = (2.5668)(1)(9.81) = 25.1801N$$

b) $$\rho_{moon} = \rho_{earth} = 2.5668\frac{kg}{m^3}$$

$$W_{moon} = \frac{g_{moon}}{g_{earth}}W_{earth} = \frac{1}{6}W_{earth} = 4.1967N$$

Problem 2. – (a) Show that p/ρ has units of velocity squared. (b) Show that p/ρ has the same units as h (kJ/kg). (c) Determine the units conversion factor that must be applied to kinetic energy, $V^2/2$, (m^2/s^2) in order to add this term to specific enthalpy h (kJ/kg).

a)

$$\left(p \approx \frac{N}{m^2}\right), \left(\rho \approx \frac{kg}{m^3}\right)$$

$$\frac{p}{\rho} \approx \left(\frac{N}{m^2}\right)\left(\frac{m^3}{kg}\right) = \frac{N-m}{kg}\left(1\frac{kg\cdot m}{N\cdot s^2}\right) = \frac{m^2}{s^2} \approx V^2$$

b)

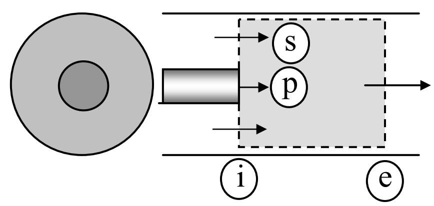
$$\frac{P}{\rho} \approx \frac{N\cdot m}{kg}\left(1\frac{J}{N\cdot m}\right)\left(\frac{1kJ}{1000\,J}\right) = \frac{1}{1000}\frac{kJ}{kg}$$

c)

$$\frac{V^2}{2} \approx \frac{m^2}{s^2}\left(1\frac{N\cdot s^2}{kg\cdot m}\right)\left(1\frac{J}{N\cdot m}\right)\left(\frac{1\,kJ}{1000\,J}\right) \approx \frac{kJ}{kg}$$

$$\therefore \text{factor} = \frac{1}{1000 g_c}$$

Problem 3. – Air flows steadily through a circular jet ejector, refer to Figure 1.15. The primary jet flows through a 10 cm diameter tube with a velocity of 20 m/s. The secondary flow is through the annular region that surrounds the primary jet. The outer diameter of the annular duct is 30 cm and the velocity entering the annulus is 5 m/s. If the flows at both the inlet and exit are uniform, determine the exit velocity. Assume the air speeds are small enough so that the flow may be treated as an incompressible flow, i.e., one in which the density is constant.



$$\dot{m}_i = \dot{m}_e$$

$$\dot{m}_i = \dot{m}_p + \dot{m}_s = \rho A_p V_p + \rho A_s V_s$$

$$\dot{m}_e = \rho A_e V_e$$

$$\therefore A_p V_p + A_s V_s = A_e V_e$$

So

$$V_e = \frac{A_p V_p + A_s V_s}{A_e}$$

$$A_e = A_s + A_p$$

$$A_p = \frac{\pi}{4}D_p^2 \qquad A_s = \frac{\pi}{4}D_o^2 - \frac{\pi}{4}D_p^2 \qquad A_e = \frac{\pi}{4}D_o^2$$

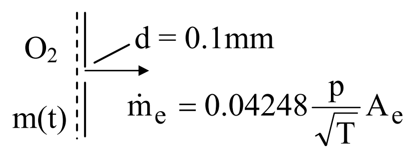
$$V_e = \frac{A_p V_p + A_s V_s}{A_e} = \frac{D_p^2 V_p + \left(D_o^2 - D_p^2\right)V_s}{D_o^2} = V_s + \frac{D_p^2}{D_o^2}\left(V_p - V_s\right)$$

$$= 5 + \frac{10^2}{30^2}\left(20 - 5\right) = 6.6667\,\text{m/s}$$

Problem 4. – A slow leak develops in a storage bottle and oxygen slowly leaks out. The volume of the bottle is 0.1 m^3 and the diameter of the hole is 0.1 mm. The initial pressure is 10 MPa and the temperature is 20˚C. The oxygen escapes through the hole according to the relation

$$\dot{m}_e = 0.04248 \frac{p}{\sqrt{T}} A_e$$

where p is the tank pressure and T is the tank temperature. The constant 0.04248 is based on the gas constant and the ratio of specific heats of oxygen. The units are: pressure N/m^2, temperature K, area m^2 and mass flow rate kg/s. Assuming that the temperature of the oxygen in the bottle does not change with time, determine the time it takes to reduce the pressure to one half of its initial value.



$$\forall = 0.1\ \text{m}^3$$
$$p_1 = 10\,\text{MPa}$$
$$T_1 = 293\,\text{K} = T_2$$
$$p_2 = 5\,\text{MPa}$$
$$R = \frac{8{,}314.3}{32} = 259.8219 \frac{\text{J}}{\text{kg} \cdot \text{K}}$$

From the continuity equation

$$\frac{dm}{dt} = -\dot{m}_e$$

but

$$m = \frac{p\forall}{RT}$$

so

$$\frac{dm}{dt} = \frac{\forall}{RT}\frac{dp}{dt} = -\dot{m}_e = -\frac{0.04248\, A_e}{\sqrt{T}} p$$

Integrating we get,

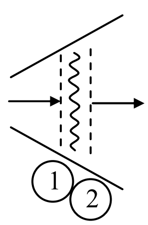
$$\ln\frac{p_2}{p_1} = -\left(\frac{0.04248\,R\sqrt{T}A_e}{\forall}\right)t$$

$$t = -\frac{\forall}{(0.04248)A_e R\sqrt{T}}\ln\frac{p_2}{p_1}$$

$$= -\frac{0.1}{(0.04248)\left(\frac{\pi}{4}\right)\left(0.1\,\text{mm}\frac{\text{m}}{1000\,\text{mm}}\right)^2(259.8219)\sqrt{293}}\ln\left(\frac{1}{2}\right)$$

$$= 46{,}713.4076\,\text{sec} = 12.9759\,\text{hrs}$$

Problem 5. – A normal shock wave occurs in a nozzle in which air is steadily flowing. Because the shock has a very small thickness, changes in flow variables across the shock may be assumed to occur without change of cross-sectional area. The velocity just upstream of the shock is 500 m/s, the static pressure is 50 kPa and the static temperature is 250 K. On the downstream side of the shock the pressure is 137 kPa and the temperature is 343.3 K. Determine the velocity of the air just downstream of the shock.



$V_1 = 500\,\text{m/s}$ $\quad$ $V_2 = ?$

$p_1 = 50\,\text{kPa}$ $\quad$ $p_2 = 137\,\text{kPa}$

$T_1 = 250\,\text{K}$ $\quad$ $T_2 = 343.3\,\text{K}$

$A_1 = A_2$

From the continuity equation

$$\dot{m}_1 = \dot{m}_2$$

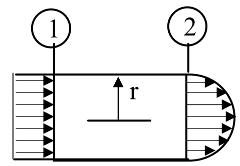
So

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$V_2 = \frac{\rho_1}{\rho_2}V_1 = \frac{p_1/RT_1}{p_2/RT_2}V_1 = \frac{p_1}{p_2}\frac{T_2}{T_1}V_1 = \left(\frac{50}{137}\right)\left(\frac{343.3}{250}\right)(500) = 250.5839\,\text{m/s}$$

Problem 6. – A gas flows steadily in a 2.0 cm diameter circular tube with a uniform velocity of 1.0 cm/s and a density ρ_o. At a cross section farther down the tube, the velocity distribution is given by $V = U_o[1-(r/R)^2]$, with r in centimeters. Find U_o, assuming the gas density to be $\rho_o[1+(r/R)^2]$.

$$V_1 = 1\,\mathrm{cm/s} \qquad V_2 = U_o\left[1-\left(\frac{r}{R}\right)^2\right]$$



$$\rho_1 = \rho_o \qquad \rho_2 = \rho_o\left[1+\left(\frac{r}{R}\right)^2\right]$$

$$\dot{m}_1 = \dot{m}_2$$

$$\dot{m}_1 = \int_o^R \rho_1 V_1 dA = \int_o^R \rho_1 V_1 2\pi r dr = \rho_o V_1 \pi R^2 = \pi R^2 \rho_o$$

$$\dot{m}_2 = \int_o^R \rho_2 V_2 dA = \int_o^R \rho_o\left(1+\frac{r^2}{R^2}\right)U_o\left(1-\frac{r^2}{R^2}\right)2\pi r dr$$

$$= \rho_o U_o 2\pi R^2 \int_o^1 \left(\xi^2 - \xi^5\right)d\xi \quad \text{where } \xi = \frac{r}{R}$$

$$= 2\pi\rho_o U_o R^2\left(\frac{1}{2}-\frac{1}{6}\right) = \frac{2}{3}\pi R^2 \rho_o U_o$$

$$\therefore \pi R^2 \rho_o = \frac{2\pi}{3} R^2 \rho_o U_o$$

so $\quad V_o = \frac{3}{2}\,\mathrm{cm/s}$

Problem 7. – For the rocket shown in Figure 1.6, determine the thrust. Assume that exit plane pressure is equal to ambient pressure.

$$\mathbf{T} = (p_e - p_{atm})A_e + \dot{m}_e V_e = 0 + (\dot{m}_H + \dot{m}_o)\left(\frac{\dot{m}_H + \dot{m}_o}{\rho_e V_e}\right) = \frac{(\dot{m}_H + \dot{m}_o)^2}{\rho_e A_e}$$

Problem 8. – Determine the force F required to push the flat plate of Figure Pl.8 against the round air jet with a velocity of 10 cm/s. The air jet velocity is 100 cm/s, with a jet diameter of 5.0 cm. Air density is 1.2 kg/m^3.

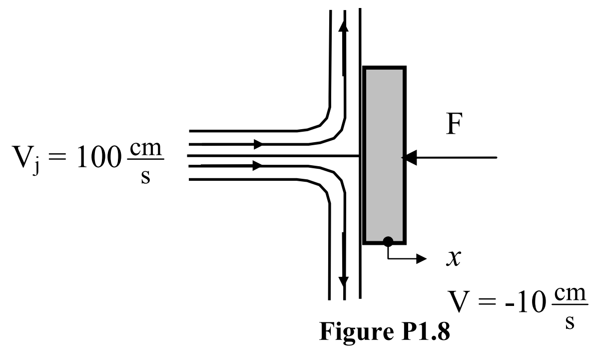
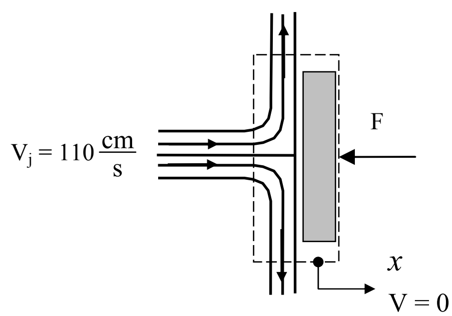


Figure P1.8



To obtain steady state add + V_p to all velocities

$$F = \dot{m}V$$

$$\dot{m} = \rho AV = (1.2)\left(\frac{\pi}{4}\right)(0.5)^2(1+0.1) = 0.002592\ \text{kg/s}$$

$$F = (0.002592)(1.1) = 0.002851\ \text{N}$$

Problem 9. – A jet engine (Figure P1.9) is traveling through the air with a forward velocity of 300 m/s. The exhaust gases leave the nozzle with an exit velocity of 800 m/s with respect to the nozzle. If the mass flow rate through the engine is 10 kg/s, determine the jet engine thrust. Exit plane static pressure is 80 kPa, inlet plane static pressure is 20 kPa, ambient pressure surrounding the engine is 20 kPa, and the exit plane area is 4.0 m^2.

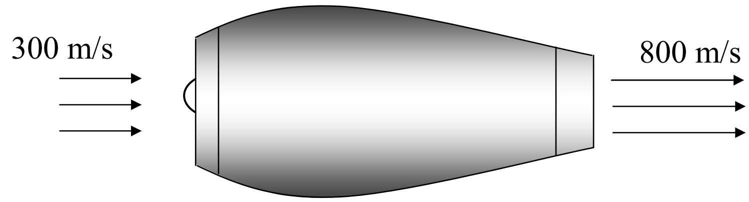


Figure P1.9

$$\mathbf{T} = (p_e - p_{atm})A_e + \dot{m}(V_e - V_i) = (80-20)(4) + (10)(800-300) = 240 + 5 = 245\text{kN}$$

Problem 10. – A high-pressure oxygen cylinder, typically found in most welding shops, accidentally is knocked over and the valve on top of the cylinder breaks off. This creates a hole with a cross-sectional area of 6.5 x 10^{-4} m^2. Prior to the accident, the internal pressure of the oxygen is 14 MPa and the temperature is 27˚C. Based on critical flow calculations, the velocity of the oxygen exiting the cylinder is estimated to be 300 m/s, the exit pressure 7.4 MPa and the exit temperature 250 K. How much thrust does the oxygen being expelled from the cylinder generate? What percentage is due to the pressure difference? What percentage due to the exiting momentum? Atmospheric pressure is 101 kPa. Also note that 0.2248 lb_f = 1 N.

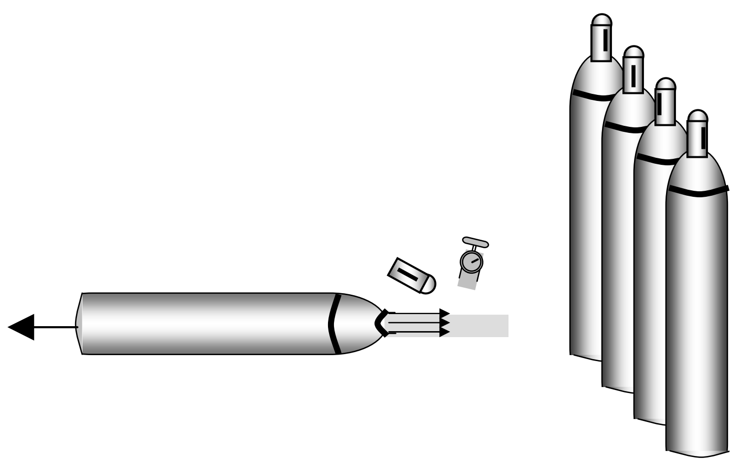


Figure P1.10

$$V_e = 300\,\text{m/s} \qquad A_e = 6.5\times10^{-4}\,\text{m}^2$$

$$p_e = 7.4\,\text{MPa} \qquad p_{atm} = 101\,\text{kPa} = 0.101\,\text{MPa}$$

$$T_e = 250\,\text{k} \qquad \dot{m} = \rho_e A_e V_e = \frac{p_e}{RT_e}A_e V_e$$

$$R = 259.82\frac{\text{J}}{\text{kg}\cdot\text{k}} \qquad = \frac{\left(7.4\times10^6\,\frac{\text{N}}{\text{m}^2}\right)}{(259.82)(250)}\left(6.5\times10^{-4}\right)(300) = 22.2\,\text{kg/s}$$

$$\mathbf{T} = \left(p_e - p_{atm}\right)A_e + \dot{m}V_e$$

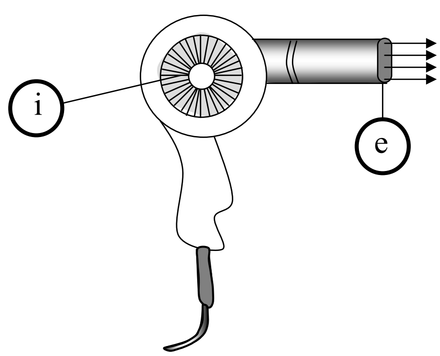
$$= \left(7400 - 101\right)\times 10^3 \times 6.4\times 10^{-4}\,\text{N} + (22.2)(300)\left(\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)\left(1\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right)$$

$$= \left(4671.4 + 6664.6\right)\text{N}$$

$$= 11{,}336.0\,\text{N} = 2{,}548.3\,\text{lb}_f$$

The thrust due to the pressure is 41% of total and that due to momentum 59%.

Problem 11. – Air enters a hand held hair dryer with a velocity of 3 m/s at a temperature of 20°C and a pressure of 101 kPa. Internal resistance heaters warm the air and it exits through an area of 20 cm^2 with a velocity of 10 m/s at a temperature of 80°C. Assume that internal obstructions do not appreciably affect the pressure between inlet and exit and that heat transfer to the surroundings are negligible. Determine the power in kW needed to operate the hair dryer at steady state.



$$\dot{m} = \rho_2 A_2 V_2 = \left(\frac{p_2}{RT_2}\right)(A_2)V_2 = \frac{(101)\left(10^3\right)}{(287)(353)}(20)\frac{\text{m}^2}{(100)^2}(10) = 0.019939\frac{\text{kg}}{\text{s}}$$

$$h_2 - h_1 = c_p\left(T_2 - T_1\right) = 1.005\frac{\text{kJ}}{\text{kg}\cdot\text{K}}(80 - 20) = 60.3\frac{\text{kJ}}{\text{kg}}$$

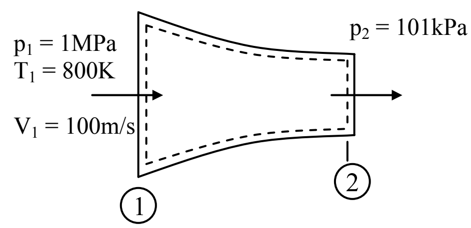
$$\frac{V_2^2 - V_1^2}{2}\left(\frac{1}{1000\,g_c}\right) = \frac{(10+3)(10-3)}{2000} = 0.0455\frac{\text{kJ}}{\text{kg}}$$

$$\dot{Q} - \dot{W} = \dot{m}\left(h_2 + \frac{V_2^2}{2}\right) - \dot{m}\left(h_1 + \frac{V_1^2}{2}\right)$$

$$-\dot{W} = \dot{m}\left[(h_2 - h_1) + \left(\frac{V_2^2 - V_1^2}{2}\right)\right] = (0.019939)(60.3 + 0.0455)$$

$$= 1.203205\,\frac{kJ}{kg} = 1{,}203.2051 W$$

Problem 12. – Air is expanded isentropically in a horizontal nozzle from an initial pressure of 1.0 MPa, of a temperature of 800 K, to an exhaust pressure of 101 kPa. If the air enters the nozzle with a velocity of 100 m/s, determine the air exhaust velocity. Assume the air behaves as a perfect gas, with R = 0.287 kJ/kg · K and γ = 1.4. Repeat for a vertical nozzle with exhaust plane 2.0 m above the intake plane.



(a) *Horizontal nozzle*

$$h_o = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$V_2 = \sqrt{V_1^2 + 2\,c_p(T_1 - T_2)} = \sqrt{V_1^2 + \frac{2\gamma R}{\gamma - 1}(T_1 - T_2)}$$

$$R = 287\,\frac{J}{kg \cdot k}$$

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{101}{1000}\right)^{\frac{.4}{1.4}} = 0.5194$$

$$T_2 = 415.5\,K$$

$$V_2 = \sqrt{(100)^2 + \frac{(2)(1.4)(287)}{0.4}(800 - 415.5)} = \sqrt{10{,}000 + 772{,}460.5} = 884.568\,m/s$$

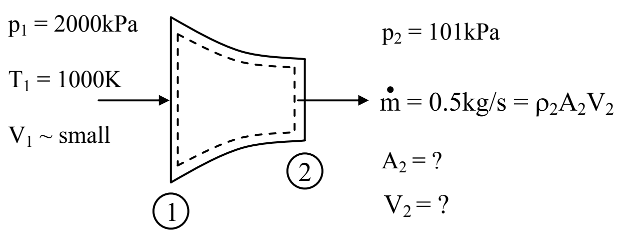
(b) *Vertical nozzle*

$$h_1 + \frac{V_1^2}{2} + gz_1 = h_2 + \frac{V_2^2}{2} + gz_2$$

$$V_2 = \sqrt{V_1^2 + \frac{2\gamma R}{\gamma - 1}(T_1 - T_2) + 2g(z_1 - z_2)} = \sqrt{782{,}460.5 + (2)(9.81)(2)}$$

$$= \sqrt{782{,}499.74} = 884.590\ \mathrm{m/s}$$

Problem 13. – Nitrogen is expanded isentropically in a nozzle from a pressure of 2000 kPa, at a temperature of 1000 K, to a pressure of 101 kPa. If the velocity of the nitrogen entering the nozzle is negligible, determine the exit nozzle area required for a nitrogen flow of 0.5 kg/s. Assume the nitrogen to behave as a perfect gas with constant specific heats, mean molecular mass of 28.0, and γ = 1.4.



$$h_o = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$V_2 = \sqrt{2(h_1 - h_2)} = \sqrt{2\,c_p(T_1 - T_2)}$$

$$T_2 = T_1\left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = (1000)\left(\frac{101}{2000}\right)^{\frac{0.4}{1.4}} = 426.1\mathrm{K}$$

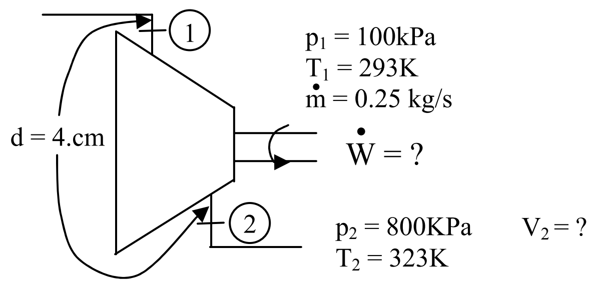
$$R = \frac{8314.3}{28} = 296.9\,\frac{\mathrm{J}}{\mathrm{kg \cdot K}}$$

$$V_2 = \sqrt{2\frac{\gamma R}{\gamma - 1}(T_1 - T_2)} = \sqrt{7R(T_1 - T_2)} = \sqrt{(7)(296.9)(1000 - 426.1)} = 1092.2\mathrm{m/s}$$

$$\rho_2 = \frac{p_2}{RT_2} = \frac{101{,}000}{(296.9)(426.1)} = 0.798\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

$$A_2 = \frac{\dot{m}}{\rho_2 V_2} = \frac{0.5}{(0.798)(1092.2)} = 0.0005734\,\text{m}^2 = 5.734\,\text{cm}^2$$

Problem 14. – Air enters a compressor with a pressure of 100 kPa and a temperature 20°C; the mass flow rate is 0.25 kg/s. Compressed air is discharged from the compressor at 800 kPa and 50°C. Inlet and exit pipe diameters are 4.0 cm. Determine the exit velocity of the air at the compressor outlet and the compressor power required. Assume an adiabatic, steady, flow and that the air behaves as a perfect gas with constant specific heats; c_p = 1.005 kJ/kg · K and R = 0.287kJ/kg·K.



$$c_p = 1.005\frac{\text{kJ}}{\text{kg}} \quad R = 0.287\frac{\text{kJ}}{\text{kg}\cdot\text{k}}$$

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = 0.25\,\text{kg/s}$$

$$A_1 = A_2 = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.04)^2 = 0.00126\,\text{m}^2$$

$$\rho_1 = \frac{p_1}{RT_1} = \frac{10.0\,\text{kPa}}{(0.287)(293)} = 1.189\frac{\text{kg}}{\text{m}^3}$$

$$\rho_2 = \frac{p_2}{RT_1} = \frac{800}{(0.287)(323)} = 8.630\frac{\text{kg}}{\text{m}^3}$$

$$V_1 = \frac{\dot{m}}{\rho_1 A_1} = \frac{0.25}{(1.189)(.00126)} = 167.3\frac{\text{m}}{\text{s}}$$

$$V_2 = \frac{\dot{m}}{\rho_2 A_2} = \frac{0.25}{(8.63)(.00126)} = 23.1\frac{\text{m}}{\text{s}}$$

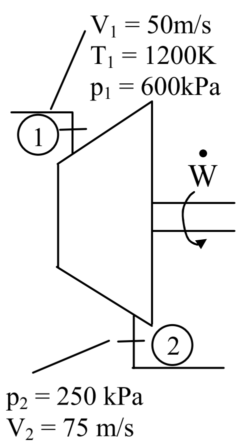
$$\dot{Q} - \dot{W} = \dot{m}_2\left(h_2 + \frac{V_2^2}{2}\right) - \dot{m}_1\left(h_1 + \frac{V_1^2}{2}\right)$$

$$-\dot{W} = \dot{m}\left[\left(h_2 - h_1\right) + \frac{V_2^2 - V_1^2}{2}\right]$$

$$= (0.25)\left[(1.005)(323 - 293) + \frac{(167.3 + 23.1)(23.1 - 1673)}{(2)(1000)}\right]$$

$$= (0.25)(30.15 - 13.73) = 4.1\frac{\text{kJ}}{\text{s}} = 4.1\,\text{kW}$$

Problem 15. – Hot gases enter a jet engine turbine with a velocity of 50 m/s*,* a temperature of 1200 K, and a pressure of 600 kPa. The gases exit the turbine at a pressure of 250 kPa and a velocity of 75 m/s. Assume isentropic steady flow and that the hot gases behave as a perfect gas with constant specific heats (mean molecular mass 25, γ = 1.37). Find the turbine power output in kJ/(kg of mass flowing through the turbine).



$$R = \frac{8314.3}{25} = 332.6\frac{\text{J}}{\text{kg}\cdot\text{K}} \qquad \gamma = 1.37 \qquad c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.37)(332.6)}{.37} = 1.2314\frac{\text{kJ}}{\text{kg}\cdot\text{K}}$$

$$T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1200\left(\frac{250}{600}\right)^{\frac{0.37}{1.37}} = 947.3\,\text{K}$$

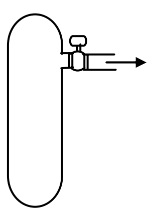
$$\dot{Q} - \dot{W} = \dot{m}_2\left(h_2 + \frac{V_2^2}{2}\right) - \dot{m}_1\left(h_1 + \frac{V_1^2}{2}\right)$$

$$\dot{W} = \dot{m}\left[(h_1 - h_2) + \frac{V_1^2 - V_2^2}{2}\right]$$

$$W = \frac{\dot{W}}{\dot{m}} = h_1 - h_2 + \frac{V_1^2 - V_2^2}{2} = c_p(T_1 - T_2) + \frac{(V_1 + V_2)(V_1 - V_2)}{2000}$$

$$W = (1.2314)(1200 - 947.3) + \left(\frac{(125)(-25)}{2000}\right) = 311.18 - 1.563 = 309.6\,\frac{kJ}{kg}$$

Problem 16. – Hydrogen is stored in a tank at 1000 kPa and 30°C. A valve is opened, which vents the hydrogen and allows the pressure in the tank to fall to 200 kPa. Assuming that the hydrogen that remains in the tank has undergone an isentropic process, determine the amount of hydrogen left in the tank. Assume hydrogen is a perfect gas with constant specific heats; the ratio of specific heats is 1.4, and the gas constant is 4.124 kJ/kg · K. The tank volume is 2.0m^3.

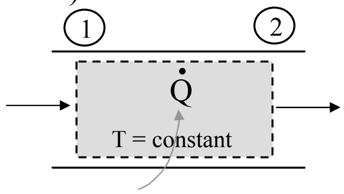


$p_1 = 1000\,\text{kPa} \qquad T_1 = 303\text{K}$

$p_2 = 200\,\text{kPa} \qquad T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = (303)\left(\frac{200}{1000}\right)^{\frac{0.4}{1.4}} = 191.3\text{K}$

$$m_2 = \frac{p_2\forall}{RT_2} = \frac{(200)(2)}{(4.124)(191.3)} = 0.507\,\text{kg}$$

Problem 17. – Methane enters a constant-diameter, 3 cm duct at a pressure of 200 kPa, a temperature of 250 K, and a velocity of 20 m/s. At the duct exit, the velocity reaches 25 m/s. For isothermal steady flow in the duct, determine the exit pressure, mass flow rate, and rate at which heat is added to the methane. Assume methane behaves as a perfect gas; the ratio of specific heats is 1.32 (constant) and the mean molecular mass is 16.0.



d = 3cm
$\gamma = 1.32$
MW = 16

$p_1 = 200$kPa
$T_1 = 250$K
$V_1 = 20$ m/s

$p_2 = ?$
$\dot{m} = ?$
$V_2 = 25$m/s
$\dot{Q} = ?$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$\left(\frac{p_1}{RT_1}\right)V_1 = \left(\frac{p_2}{RT_2}\right)V_2$$

$$p_1 V_1 = p_2 V_2$$

$$p_2 = p_1\left(\frac{V_1}{V_2}\right) = (200)\left(\frac{20}{25}\right) = 160\,\frac{N}{m^2}$$

$$A = \frac{\pi}{4}(0.03)^2 = 0.000707\text{m}^2$$

$$R = \frac{8314.3}{16} = 519.6\,\frac{J}{kg \cdot K}$$

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)(A_1)V_1 = \frac{(200)}{(519.6)}\frac{(0.000707)}{250}(20) = 0.02176\,\frac{kg}{s}$$

$$\dot{Q} = \dot{m}\left(\frac{V_2^2 - V_1^2}{2}\right) = (0.02176)\frac{(25+20)(5)}{2} = 2.448\text{ W}$$

Problem 18. – Air is adiabatically compressed from a pressure of 300 kPa and a temperature of 27 C to a pressure of 600 kPa and a temperature of 327 C. Is this compression actually possible?

$p_1 = 300\,\text{kPa} \qquad\qquad p_2 = 600\,\text{kPa}$

$T_1 = 27 + 273 = 300\,\text{K} \qquad\qquad T_2 = 327 + 273 = 600\,\text{K}$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = c_p \ln\frac{600}{300} - R\ln\frac{600}{300}$$

$$= \left(c_p - R\right)\ln 2 = c_v \ln 2 > 0 \quad \therefore \text{possible}$$

Problem 19. – Two streams of air mix in a constant-area mixing tube of a jet ejector. The primary jet enters the tube with a speed of 600 m/s, a pressure of 200 kPa and a temperature of 400˚C. The secondary stream enters with a velocity of 30 m/s, a pressure of 200 kPa and a temperature of 100˚C. The ratio of the area of the secondary flow to the primary jet is 5:1. The air behaves as a perfect gas with constant specific heats, c_p = 1.0045 kJ/kg· K. Using the iterative numerical procedure described in Example 1.9 determine the velocity, pressure and temperature of the air leaving the mixing tube.

g_c	1	
α	5	
γ	1.4	
R	287	
c_p	1004.5	
	Primary	**Secondary**
V	600.00	30.00
T	673	373
P	200,000	200,000

A	43,122.5078
B	263,528.7595
C	706,538.5693

n	V_e (m/s)	P_e (Pa)	T_e (K)
1	0.0000	101,000.0	293.1500
2	125.1620	244,722.8	695.5757
3	122.5671	245,112.7	695.8957
4	122.4284	245,133.6	695.9126
5	122.4210	245,134.7	695.9135
6	122.4206	245,134.7	695.9136
7	122.4206	245,134.7	695.9136

Problem 20. – The flow exiting a jet ejector was determined by utilizing an iterative numerical procedure. A more direct approach is possible however. Eliminate pressure P_e between Eqs. (1.53) and (1.54). Solve for the temperature T_e in the resulting expression, and equate it to Eq. (1.55). This produces a quadratic equation for the velocity V_e. Solve the quadratic to determine V_m for the same set of conditions given in Example 1.9.

From Eq. (1.53),

$$p_e = \frac{AT_e}{V_e}$$

From Eq. (1.54),

$$p_e = B - \frac{A}{R}V_e$$

Combine these to obtain

$$T_e + \frac{V_e^2}{R} = \frac{B}{A}V_e$$

Equation (1.55) can be written as

$$T_e + \frac{1}{2c_{p_e}}V_e^2 = \frac{C}{c_{p_e}}$$

Eliminate T_e to obtain the quadratic

$$aV_e^2 - bV_e + c = 0$$

where

$$a = \left(\frac{1}{R} - \frac{1}{2c_{p_e}}\right)$$

$$b = \frac{B}{A}$$

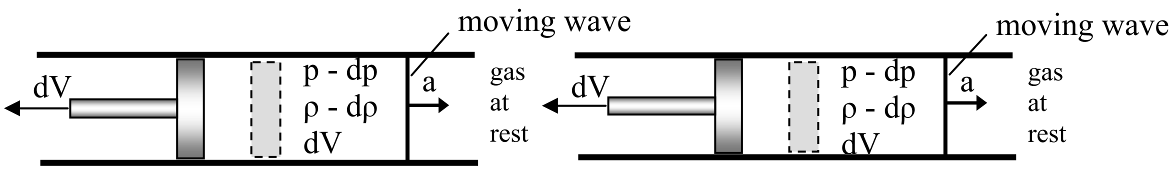
$$c = \frac{C}{c_{p_e}}$$

$$\therefore V_e = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chapter Two

WAVE PROPAGATION IN COMPRESSILBE MEDIA

Problem 1. – Using the expansion wave and control volume depicted in Figs. 2.8 and 2.9 along with the continuity and momentum equations, rederive Eq. (2.4).



Continuity equation

$$(\rho - d\rho)(a + dV)A - \rho a A = 0$$

Expand, neglect products of derivatives and simplify to get

$$\rho dV - a d\rho = 0 \qquad (1)$$

Momentum equation

$$pA - (p - dp)A = \rho a A[(a + dV) - a]$$

or

$$dp = \rho a dV \qquad (2)$$

Combining Eqs. (1) and (2) gives

$$dp = a^2 d\rho$$

Since the process is reversible and adiabatic, i.e., isentropic, this can be written as:

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

Problem 2. – (a) Derive an expression for k_s, for a perfect gas, substitute the result into Eq. (2.10), and thereby demonstrate Eq. (2.7); (b) Derive an expression for k_T, for a perfect gas, substitute the result into Eq. (2.11), and thereby demonstrate Eq. (2.7) and finally; (c) Derive an expression for β_s, for a perfect gas, substitute your result into Eq. (2.14), and thereby demonstrate Eq. (2.7).

(a) $$k_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$$

An isotropic process involving a perfect gas is described by $P = c\rho^{\gamma}$

$$\therefore \quad \frac{dp}{d\rho} = \gamma c \rho^{\gamma-1} = \frac{\gamma c \rho^{\gamma}}{\rho} = \frac{\gamma p}{\rho}$$

Hence,

$$\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{\rho}{\gamma p}$$

$$k_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s = \frac{1}{\gamma p} = \frac{1}{\gamma \rho RT}$$

So,

$$a = \sqrt{\frac{1}{\rho k_s}} = \sqrt{\gamma RT}$$

(b) $$k_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$$

$$\rho = \frac{p}{RT}$$

$$\left(\frac{\partial \rho}{\partial p}\right)_T = \frac{1}{RT}$$

$$k_T = \frac{1}{\rho RT}$$

So,

$$a = \sqrt{\frac{\gamma}{\rho k_T}} = \sqrt{\gamma RT}$$

(c) $$\beta_s = \rho\left(\frac{\partial p}{\partial \rho}\right)_s = \rho\left(\frac{\gamma p}{\rho}\right) = \gamma p = \gamma\rho RT$$

$$a = \sqrt{\frac{\beta_s}{\rho}} = \sqrt{\gamma RT}$$

Problem 3. – Use dimensional analysis to develop an expression for the speed of sound in terms of the isentropic compressibility, the density and g_c.

$$a = f(k_s, \rho, g_e)$$

$$a \sim \frac{L}{T}, \quad k_s \sim \frac{L^2}{F}, \quad \rho \sim \frac{M}{L^3}, \quad g_c \sim \frac{ML}{FT^2}$$

$$\frac{L}{T} = \left(\frac{L^2}{F}\right)^a \left(\frac{M}{L^3}\right)^b \left(\frac{ML}{FT^2}\right)^c = F^{-a-c} M^{b+c} L^{2a-3b+c} T^{-2c}$$

$$F: -a - c = 0$$

$$M: b + c = 0$$

$$L: 2a - 3b + c = 1$$

$$T: -2c = 1$$

Hence, $c = \frac{1}{2} \qquad a = -\frac{1}{2} \qquad b = -\frac{1}{2}$

So,

$$a = \sqrt{\frac{g_c}{\rho k_s}}$$

Problem 4. – Using the data provided in Tables 2-1, 2-2 and 2-3, i.e., the density, and the isentropic compressibility or the bulk modulus, calculate the velocity of sound at 20°C and one atmosphere pressure in (a) helium, (b) turpentine, and (c) lead.

(a) *Helium*: $\rho = 0.16\frac{kg}{m^3}, \; k_s = 5{,}919\frac{1}{GPa}$

$$a = \sqrt{\frac{1}{\rho k_s}} = \sqrt{\frac{10^9}{(0.16)(5{,}919)}} = 1027.6\ \mathrm{m/s}$$

(b) *Turpentine*: $\rho = 870\dfrac{\mathrm{kg}}{\mathrm{m}^3}$, $k_s = 0.736\dfrac{1}{GPa}$

$$a = \sqrt{\frac{1}{\rho k_s}} = \sqrt{\frac{10^9}{(870)(0.736)}} = 1249.7\ \mathrm{m/s}$$

(c) *Lead*: $\rho = 11{,}300\dfrac{\mathrm{kg}}{\mathrm{m}^3}$, $\beta_s = 16.27\ \mathrm{GPa}$

$$a = \sqrt{\frac{\beta_s}{\rho}} = \sqrt{\frac{(16.27)\,10^9}{11{,}300}} = 1199.9\ \mathrm{m/s}$$

Problem 5. – In Example Problem 2.3 the speed of sound of superheated steam was determined by using a finite difference representation of the compressibility and steam table data (Table 2-4). Using the same steam table data, determine the speed of sound of superheated steam for the same pressure and temperature, i.e., at p = 500 kPa and T = 300˚C. However, use the following finite differences to obtain two estimates for the speed of sound:

$$a^2 = \frac{\gamma}{\left(\dfrac{\partial \rho}{\partial p}\right)_T} = \frac{\gamma}{\left[\dfrac{\partial (1/v)}{\partial p}\right]_T}$$

$$a^2 = \frac{1}{\left(\dfrac{\partial \rho}{\partial p}\right)_s} = \frac{1}{\left[\dfrac{\partial (1/v)}{\partial p}\right]_s}$$

$$a^2 = \frac{\gamma}{\left[\dfrac{\partial (1/v)}{\partial p}\right]_T} = \frac{\gamma}{\dfrac{\dfrac{1}{v(p+\Delta p, T)} - \dfrac{1}{v(p-\Delta p, T)}}{2\Delta p}} = \frac{2\gamma \Delta p}{\dfrac{1}{v(p+\Delta p, T)} - \dfrac{1}{v(p-\Delta p, T)}}$$

From Example 2.3

$$v(p + \Delta p_1 T) = 0.4344\frac{M^3}{\mathrm{kg}}, \quad v(p - \Delta p_1 T) = 0.6548\frac{M^3}{\mathrm{kg}}, \text{ and } \Delta p = 100{,}000\ \mathrm{Pa}$$

$$a^2 = \frac{(2)(1.327)(100{,}000)}{\frac{1}{0.4344} - \frac{1}{0.6548}} = 342{,}521.5 \frac{m^2}{s^2}$$

$$a = 585.3 \text{ m/s}$$

$$a^2 = \frac{1}{\left[\frac{\partial(1/v)}{\partial p}\right]_s} = \frac{2\Delta p}{\frac{1}{v(p+\Delta p, s)} - \frac{1}{v(p-\Delta p, s)}}$$

From Example 2.3

$$v(p+\Delta p, s) = 0.4544 \frac{M^3}{kg}, \quad v(p-\Delta p, s) = 0.6209 \frac{M^3}{kg} \text{ and } \Delta p = 100{,}000 \text{ Pa}$$

$$a^2 = \frac{(2)(100{,}000)}{\frac{1}{0.4544} - \frac{1}{0.6209}} = 338{,}903.2 \frac{m^2}{s^2}$$

$$a = 582.2 \text{ m/s}$$

Problem 6. – Equation (2.16) provides a convenient expression for calculating the speed of sound in air: a = $20.05\sqrt{T}$, where T is the absolute temperature in degrees Kelvin. Derive the following linear equation for the speed of sound in air:

$$a = a_0 + 0.6t$$

where a_0 is the speed of sound in air at 0°C and t is °C.

To accomplish this make use of Eq. (2-16) and the expansion

$$(x+y)^n = x^n + nx^{n-1}y + ...$$

$$a = \sqrt{\gamma RT} = [\gamma R(273+t)]^{1/2}$$

$$= \sqrt{\gamma R(273)}\left(1 + \frac{t}{273}\right)^{1/2} = a_o\left(1 + \frac{1}{2}\frac{t}{273} + ...\right)$$

$$a_o = \sqrt{\gamma R(273)} = 20.05\sqrt{273} = 331\frac{m}{s},$$

$$\frac{a_o}{(2)(273)} = \frac{331}{546} = 0.6$$

$$\therefore \quad a = 331 + 0.6t$$

Problem 7. – Rather than measure the bulk modulus directly it may be easier to measure the speed of sound as it propagates though a material and then use it to compute the bulk modulus. For a Lucite plastic of density 1,200 kg/m^3, the speed of sound is measured as 2,327 m/s. Determine the bulk modulus. What is the corresponding isentropic compressibility?

Now $\rho = 1{,}200\frac{kg}{m^3}$, $a = 2{,}327\frac{m}{s}$

$$a = \sqrt{\frac{\beta_s}{\rho}}$$

so, $$\beta_3 = \rho a^2 = \left(1{,}200\frac{kg}{ms}\right)\left(2{,}327\frac{m}{s}\right)^2\left(1\frac{N\cdot s^2}{kg\cdot m}\right) = 6.498\times10^9\,Pa = 6.498\,GPa$$

$$k_s = \frac{1}{\beta_s} = 0.1539\frac{1}{GPa}$$

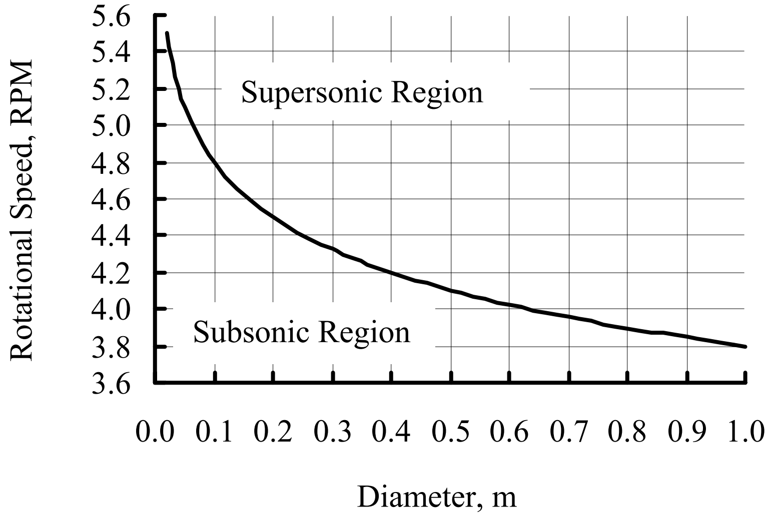
Problem 8. – An object of diameter d (m) is rotated in air at a speed of N revolutions per minute. Draw a plot of the rotational speed required for the velocity at the outer edge of the object to just reach sonic velocity for a given diameter. Take the speed of sound of the air to be 331m/s.

The highest speed will occur at R.

$$V = N\left(\frac{rev}{min}\right)\left(\frac{2\pi rad}{rev}\right)R(m)\frac{1\,min}{60\,s} = a = 331\frac{m}{s}$$

$$= \frac{\pi}{60}ND, \frac{m}{s}$$

The following is a log base 10 plot of N = 6,321.6/D.



Problem 9. – (a) Newton assumed that the sound wave process was isothermal rather than isentropic. Determine the size of error made in computing the speed of sound by making this assumption. (b) A flash of lightening occurs in the distance. 20 seconds later the sound of thunder is heard. The temperature in the area is 23°C. How far away was the lightening strike?

(a) $$a_s = \sqrt{\frac{1}{\rho k_s}} \qquad a_T = \sqrt{\frac{1}{\rho k_T}} \qquad \therefore \quad \frac{a_T}{a_s} = \sqrt{\frac{k_s}{k_T}}$$

$$\frac{a_T - a_s}{a_s} = \left(\frac{1}{\sqrt{\gamma}} - 1\right)100 \qquad \text{for } \gamma = 1.4 \qquad \frac{a_T - a_s}{a_s} = -15.5\%$$

(b) $$L = a\Delta t = (344.86)(20) = 6{,}897.2 \text{ m}$$

Problem 10. – (a) The pressure increase across a compression pulse moving into still air at 1 atmosphere pressure and 30°C is 100 Pa. Determine the velocity following the pulse. (b) The velocity changes by 0.1 m/s across a pressure wave that moves into hydrogen gas that is at rest at a pressure of 100 kPa and temperature 300K. Determine the pressure behind the wave.

Use Eq (2.2) and write the expression in difference form as

(a) $\Delta V = \dfrac{\Delta p}{\rho a}, \qquad \Delta p = 100 \text{ Pa}$

air: $\rho = \dfrac{101{,}000}{\left(\dfrac{8314}{28.97}\right)(303)} = 1.1615 \dfrac{kg}{m^3}$

$$a = 20.05\sqrt{303} = 349.0 \; m/s$$

Therefore, $\Delta V = \dfrac{100}{(1.1615)(349.0)} = 0.247 \; m/s$

(b) $\Delta p = \rho a \Delta V, \quad \Delta V = 0.1 \; m/s$

hydrogen: $\rho = \dfrac{p}{RT} = \dfrac{100{,}000}{\left(\dfrac{8314}{2.016}\right)(300)} = 0.0808 \dfrac{kg}{m^3}$

$$a = \sqrt{(1.41)\left(\frac{8314}{1.016}\right)(300)} = 1320.8 \text{ m/s}$$

Therefore, $\Delta p = (0.0808)(1320.8)(0.1) = 10.68 \text{ Pa}$

Problem 11. – (a) Helium at 35°C is flowing at a Mach number of 1.5. Find the velocity and determine the local Mach angle. (b) Determine the velocity of air at 40°C to produce a Mach angle of 38°

(a) *helium*: $T = 35°C = 308K \quad M = 1.5$

$$V = aM \qquad a = \sqrt{\gamma RT} = \sqrt{(1.667)\left(\frac{8{,}314}{4{,}003}\right)(308)} = 1{,}032.7 \text{ m/s}$$

$$V = (1{,}032.7)(1.5) = 1{,}549.0 \text{ m/s}$$

$$\mu = \sin^{-1}\left(\frac{1}{\mu}\right) = 41.8°$$

(b) *air*: $T = 40°C = 313K \qquad a = 20.05\sqrt{223.3} = 299.6 \text{ m/s}$

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

$$M = \frac{1}{\sin\mu} = \frac{V}{a}$$

$$V = \frac{a}{\sin\mu} = \frac{354.6}{\sin(38)} = 576.0 \text{ m/s}$$

Problem 12. – (a) A jet plane is traveling at Mach 1.8 at an altitude of 10 km where the temperature is 223.3K. Determine the speed of the plane. (b) Air at 320 K flows in a supersonic wind tunnel over a 2-D wedge. From a photograph the Mach angle is measured to be 45°. Determine the flow velocity, the local speed of sound and the Mach number of the tunnel.

(a) $M = 1.8$, $T = 223.3\text{ K}$, $a = 20.05\sqrt{223.3} = 299.6 \text{ m/s}$

$$V = aM = (299.6)(1.8) = 539.3 \text{ m/s}$$

(b) air: $T = 320\text{ K}$, $\mu = 45°$, $a = 20.05\sqrt{320} = 358.7 \text{ m/s}$

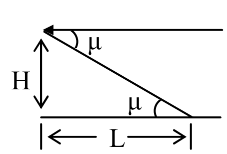
$$V = \frac{a}{\sin\mu} = \frac{358.7}{\sin(45)} = 507.2 \text{ m/s}$$

$$M = \frac{V}{a} = \frac{1}{\sin\mu} = 1.414$$

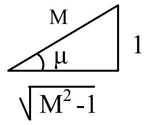
Problem 13. – A supersonic aircraft, flying horizontally a distance H above the earth, passes overhead. Δt later the sound wave from the aircraft is heard. In this time increment, the plane has traveled a distance L. Show that the Mach number of the aircraft can be computed from:

$$M = \sqrt{\left(\frac{L}{H}\right)^2 + 1} = \sqrt{\left(\frac{V\Delta t}{H}\right)^2 + 1}$$

Hint: first show that the Mach angle μ can be expressed as $\tan^{-1}\left(1/\sqrt{M^2 - 1}\right)$ and then connect the Mach angle, μ, to the geometric parameters H and L.



$$\sin\mu = \frac{1}{M}$$



$$\tan\mu = \frac{1}{\sqrt{M^2-1}} = \frac{H}{L}$$

$$\therefore\ M = \sqrt{\left(\frac{L}{H}\right)^2 + 1} \qquad \text{but}\quad L = V\Delta t$$

$$= \sqrt{\left(\frac{V\Delta t}{H}\right)^2 + 1}$$

Problem 14. – Given speeds and temperatures, determine the corresponding Mach numbers of the following (note: 1 mile = 5,280 ft = 1,609.3 m; 1 mi/hr = 1.6093 km/hr = 0.447 m/s):

(a) A cheetah running at top speed of 60 mi/hr; the local temperature is 40°C
(b) A Peregrine falcon in a dive at 217 mi/hr; local temperature of 25°C
(c) In June 1999 in Athens Greece, Maurice Greene became the world's fastest human by running 100 m in 9.79 s; the temperature was 21°C
(d) In June 1999, Alexander Popov became the world's fastest swimmer by swimming 50 m in 21.64s; the temperature of the water was 20°C

(a) $a = 20.05\sqrt{313} = 354.7\ \text{m/s}$

$$M = \frac{V}{a} = \frac{(60)\dfrac{\text{mi}}{\text{hr}}(0.447)\dfrac{\text{m/s}}{\text{mi/hr}}}{(354.7)\,\text{m/s}} = 0.076$$

(b) $a = 20.05\sqrt{298} = 346.1\ \text{m/s}$

$$M = \frac{V}{a} = \frac{(217)(0.447)}{(346.1)} = 0.28$$

(c) $a = 20.05\sqrt{294} = 343.8\ \text{m/s}$ $\qquad V = \dfrac{100}{9.79} = 10.21\ \text{m/s} \quad \left(= \dfrac{10.21}{0.447} = 22.9\dfrac{\text{mi}}{\text{hr}}\right)$

$$M = \frac{10.21}{343.8} = 0.03$$

(d) $a = 1{,}481\ \text{m/s (from Table 2 - 2)}$ $\quad V = \dfrac{50}{21.64} = 2.31\ \text{m/s} \quad \left(= \dfrac{2.31}{0.447} = 5.17\dfrac{\text{mi}}{\text{hr}}\right)$

$$M = \frac{2.31}{1{,}481} = 0.00156$$

Problem 15. – Given speeds and Mach numbers, assuming air is a perfect gas, determine the corresponding local temperature (note: 1 mi/hr = 0.447 m/s) for the following:

(a) A Boeing 747-400 at a cruise speed of 910 km/hr; M = 0.85.
(b) Concorde at a cruise speed of 1,320 mi/hr; M = 2.0
(c) The fastest airplane, the Lockheed SR-71 Blackbird, flying at 2,200 mi/hr; M = 3.3
(d) The fastest boat, the Spirit of Australia, that averaged 317.6 mi/hr; M = 0.41
(e) The fastest car, the ThrustSSC, averaged 760.035 mi/hr; M = 0.97

(a) $$V = \frac{910{,}000\text{m}}{3600\text{s}} = 252.8\frac{\text{m}}{\text{s}} \qquad M = 0.85 \qquad a = \frac{V}{M} = \frac{252.8}{.85} = 297.4\frac{\text{m}}{\text{s}}$$

$$T = \left(\frac{a}{20.05}\right)^2 = \left(\frac{297.4}{20.05}\right)^2 = 220°\text{K} = -53°\text{C}$$

(b) $$V = (1320)(0.447) = 590.0\frac{\text{m}}{\text{s}} \qquad M = 2.0 \qquad a = \frac{V}{M} = 295.0\text{ m/s}$$

$$T = \left(\frac{a}{20.05}\right)^2 = \left(\frac{295}{20.05}\right)^2 = 216.5\text{K} = -56.5°\text{C}$$

(3) $$V = (2200)(0.447) = 983.4\frac{\text{m}}{\text{s}} \qquad M = 3.3 \qquad a = \frac{983.4}{3.3} = 298.0\text{ m/s}$$

$$T = \left(\frac{a}{20.05}\right)^2 = \left(\frac{298.0}{20.05}\right)^2 = 220.9\text{K} = -52.1°\text{C}$$

(d) $$V = (317.6)(0.447) = 142.0\frac{\text{m}}{\text{s}} \qquad M = 0.41 \qquad a = \frac{142}{.41} = 346.3\text{ m/s}$$

$$T = \left(\frac{a}{20.05}\right)^2 = \left(\frac{346.3}{20.05}\right)^2 = 298.2\text{K} = 25.2°\text{C}$$

(e) $$V = (760.035)(0.447) = 339.7\text{ m/s} \qquad M = 0.97 \qquad a = \frac{339.7}{.97} = 350.2\text{ m/s}$$

$$T = \left(\frac{a}{20.05}\right)^2 = \left(\frac{350.2}{20.05}\right)^2 = 305.1\text{K} = 32.1°\text{C}$$

Problem 16. – A baseball, which has a mass of 145 grams and a diameter of 3.66 cm, when dropped from a very tall building reaches high speeds. If the building is tall enough the speed will be controlled by the drag, as the baseball will reach terminal speed. At this state

$$W = F_D$$

Where W (weight) = mg, g (acceleration of gravity) = 9.81 m/s^2, F_D (drag force) = $C_D\rho_{air}AV^2/2$, C_D (drag coefficient) = 0.5 and A (projected area of sphere) = πR^2. Find the terminal speed of the baseball and determine the corresponding Mach number if the ambient air temperature is 23°C and the ambient air pressure is 101 kPa..

The density of the air is first determined:

$$\rho_{air} = \frac{p}{RT} = \frac{101}{(0.287)(296)} = 1.19\,\mathrm{kg/m^3}$$

Now

$$W = mg = F_D = \frac{C_D A \rho_{air} V^2}{2}$$

Hence,

$$V = \sqrt{\frac{2mg}{C_D \rho_{air} A}} = \sqrt{\frac{2(0.145)(9.81)}{(0.5)(1.19)(0.0042)}} = 33.76\,\mathrm{m/s}$$

$$M = \frac{V}{a} = \frac{33.76}{\sqrt{(1.4)(287)(296)}} = 0.098$$

Problem 17. – Derive the following equation for the speed of sound of a real gas from Berthelot's equation of state:

$$p = \frac{\rho RT}{1-\beta\rho} - \frac{\alpha\rho^2}{T}$$

$$a = \sqrt{\gamma\left[\frac{RT}{1-\beta\rho} + \frac{RT\rho\beta}{(1-\beta\rho)^2} - \frac{2\alpha\rho}{T}\right]}$$

Since T is treated as a constant, we may simply use information from Section 2.6 where

$$a = \sqrt{\gamma\left(\frac{\partial p}{\partial \rho}\right)_T}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{RT}{1-\beta\rho} + \frac{RT\rho\beta}{(1-\beta\rho)^2} - 2\alpha\rho$$

Now replace α with α/T. Thus, from Eq. (2.24)

$$a = \sqrt{\gamma\left[\frac{RT}{1-\beta\rho} + \frac{RT\rho\beta}{(1-\beta\rho)^2} - \frac{2\alpha\rho}{T}\right]}$$

Problem 18. – Using the speed of sound expression from the previous problem and the following constants for nitrogen

$R = 296.82$ (N·m)/(kg·K)
$\alpha = 21{,}972.68$ N·m^4/kg^2
$\beta = 0.001378$ m^3/kg
$\gamma = 1.4$

determine the speed of sound for the two cases described in Example 2.4.

Case (1) p 0.3 MPa and T = 300K

Iteration	v	f(v)	df /dv	v-f/(df/dv)	ρ	a
1	0.296823	-4.9286E-05	8.7530E-02	0.297386	3.3690	353.7517
2	0.297386	1.8796E-07	8.8198E-02	0.297384	3.3626	353.7505
3	0.297384	2.6975E-12	8.8195E-02	0.297384	3.3627	353.7505

The result differs from the experimental value 353.47 m/s by 0.08%.

Case (2): p 30.0 MPa and T = 300K

Iteration	v	f(v)	df /dv	v-f/(df/dv)	ρ	a
1	0.002968	-8.2594E-09	3.0708E-06	0.005658	336.9016	604.3973
2	0.005658	5.2436E-08	4.9296E-05	0.004594	176.7430	426.1798
3	0.004594	1.3084E-08	2.5826E-05	0.004088	217.6647	457.9898
4	0.004088	2.2920E-09	1.7035E-05	0.003953	244.6426	483.2795
5	0.003953	1.4088E-10	1.4959E-05	0.003944	252.9695	491.8702
6	0.003944	6.6552E-13	1.4817E-05	0.003944	253.5736	492.5088
7	0.003944	1.5099E-17	1.4817E-05	0.003944	253.5765	492.5118

The result differs from the experimental value 483.18 m/s by 1.9%.

Problem 19. **–**Employ the finite difference method of Example 2.5 to determine the speed of sound in nitrogen using the Redlich-Kwong equation of state

$$p = \frac{RT\rho}{1-\beta\rho} - \frac{a_o\rho^2}{(1+\beta\rho)\sqrt{T}}$$

where for nitrogen:

$R = 296.823$ (N·m)/(kg·K)
$a_o = 1979.453$ (N·m^4·√K)/(kg^2)
$\beta = 0.0009557$ m^3/kg
$\gamma = 1.4$

Compute the speed at a pressure of 30.1 MPa and a temperature of 300 K. Experimental values of the speed of sound of nitrogen may be found in Ref. (11). For the given conditions the measured value is 483.730 m/s.

The Redlich-Kwong equation of state is: $p = \dfrac{RT}{v-\beta} - \dfrac{a_o}{v(v+\beta)\sqrt{T}}$. Rearrange to obtain:

$$f(v) = v^3 - \left(\frac{RT}{p}\right)v^2 - \left(\beta^2 + \frac{RT\beta}{p} - \frac{a_o}{p\sqrt{T}}\right)v - \frac{a_o\beta}{p\sqrt{T}} = 0$$

$$\frac{df}{dv} = 3v^2 - 2\left(\frac{RT}{p}\right)v - \left(\beta^2 + \frac{RT\beta}{p} - \frac{a_o}{p\sqrt{T}}\right)$$

Use Newton-Raphson to find v = 0.003279 m^3/kg. Thus, ρ = 304.9917 kg/m^3. Use Δρ = 0.1 and compute

$$p(\rho+\Delta\rho,T) = p(305.0917,300) = 30,112,951.62 \text{ Pa}$$
$$p(\rho-\Delta\rho,T) = p(304.8917,300) = 30,087,052.10 \text{ Pa}$$

$$a = \sqrt{\gamma \frac{\Delta p}{\Delta \rho}} = 425.79 \frac{m}{s}$$

The result is 12% too small compared to the experimental value of 483.73m/s. However, if a more appropriate value of γ at this pressure and temperature is used, i.e., $\gamma = 1.704$, a = 469.75m/s, which is in error by only 2.9%.

Chapter Three

ISENTROPIC FLOW OF A PERFECT GAS

Problem 1. – Air flows at Mach 0.25 through a circular duct with a diameter of 60 cm. The stagnation pressure of the flow is 500 kPa; the stagnation temperature is 175°C. Calculate the mass flow rate through the channel, assuming γ = 1.4 and that the air behaves as a perfect gas with constant specific heats.

$$p = \left(\frac{p}{p_o}\right) 500\text{ kPa} = 0.9575(500) = 478.7500\text{ kPa}$$

$$T = \left(\frac{T}{T_o}\right)(175 + 273) = 0.9877(448) = 442.4896\text{ K}$$

$$\rho = \frac{p}{RT} = \frac{(478.75\text{ kN/m}^2)}{(0.287\text{ kN}\cdot\text{m/kg}\cdot\text{K})(442.4896\text{ K})} = 3.7698\text{ kg/m}^3$$

$$A = \frac{\pi}{4}(0.6)^2 = 0.2827\text{ m}^2$$

$$V = M\sqrt{\gamma RT} = 0.25\sqrt{1.4(287\text{ N}\cdot\text{m/kg}\cdot\text{K})(442.4896\text{ K})} = 105.4136\text{ m/s}$$

$$\dot{m} = \rho AV = 112.3603\text{ kg/s}$$

Problem 2. – Helium flows at Mach 0.50 in a channel with cross-sectional area of 0.16 m^2. The stagnation pressure of the flow is 1 MPa, and stagnation temperature is 1000 K. Calculate the mass flow rate through the channel, with γ = 5/3.

$$p = \left(\frac{p}{p_o}\right) 1\text{ MPa} = 0.8186(1000\text{ kPa}) = 818.6\text{ kPa}$$

$$T = \left(\frac{T}{T_o}\right)(1000\text{ K}) = 0.9231(1000) = 923.1\text{ K}$$

$R = 2.077\ \text{kJ/kg}\cdot\text{K}$

$$\rho = \frac{p}{RT} = \frac{818.6}{(2.077)(923.1)} = 0.4270\ \text{kg}/\text{m}^3$$

$$V = M\sqrt{\gamma RT} = 0.50\sqrt{(5/3)(2077\ \text{N}\cdot\text{m/kg}\cdot\text{K})(923.1\,\text{K})} = 893.7931\ \text{m/s}$$

$$\dot{m} = \rho AV = (0.4270\ \text{kg}/\text{m}^3)(0.16\ \text{m}^2)(893.7931\ \text{m/s}) = 61.0639\ \text{kg/s}$$

Problem 3. – In Problem 2, the cross-sectional area is reduced to 0.12 m^2. Calculate the Mach number and flow velocity at the reduced area. What percent of further reduction in area would be required to reach Mach 1 in the channel?

$$\frac{A_2}{A^*} = \frac{A_2}{A_1}\frac{A_1}{A^*} = \left(\frac{0.12}{0.16}\right)1.3203 = 0.9902$$

So, $A_2 < A^*$ for $M_1 = 0.5$. Therefore, $M_2 = 1$ and M_1 will be reduced below 0.5. Since the exit Mach number is 1, then $A_2 = A^*$,

$$\frac{A_1}{A^*} = \frac{A_1}{A_2}\frac{A_2}{A^*} = \left(\frac{0.16}{0.12}\right)1 = 1.3333$$

Using this area ratio we find: $M_1 = 0.4930$. Now $M_2 = 1$ so

$$T_2 = \left(\frac{T_2}{T_o}\right)T_o = (0.7500)1000 = 750.0\ \text{K}$$

$$V_2 = M_2\sqrt{\gamma RT} = 1.0\sqrt{(5/3)(2077)750} = 1611.2883\ \text{m/s}$$

Problem 4. – (a) For small Mach numbers, determine an expression for the density ratio ρ/ρ_o. (b) Using Eqs. (3.15) and (3.17), prove that

$$\left(\frac{p_o}{p}\right)\left(\frac{\rho}{\rho_o}\right) = \left(\frac{T_o}{T}\right) = \left(\frac{a_o}{a}\right)^2$$

(a)

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} \cong 1 + \left(\frac{\gamma-1}{2}\right)\left(-\frac{1}{\gamma-1}\right)M^2 + \cdots = 1 - \frac{M^2}{2} + \cdots$$

(b)

$$\left(\frac{p_o}{p}\right)\left(\frac{\rho}{\rho_o}\right) = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}\left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2}M^2\right)^1$$

$$= \frac{T_o}{T} = \frac{a_o^2/\gamma R}{a^2/\gamma R} = \left(\frac{a_o}{a}\right)^2$$

Problem 5. – An airflow at Mach 0.6 passes through a channel with a cross-sectional area of 50 cm^2. The static pressure in the airstream is 50 kPa; static temperature is 298 K.

(a) Calculate the mass flow rate through the channel.

(b) What percent of reduction in area would be necessary to increase the flow Mach number to 0.8? to 1.0?

(c) What would happen if the area were reduced more than necessary to reach Mach 1?

(a) $$\rho = \frac{p}{RT} = \frac{50 \text{ kPa}}{(0.287 \text{ kN}\cdot\text{m/kg}\cdot\text{K})298 \text{ K}} = 0.5846 \text{ kg/m}^3$$

$$V = M\sqrt{\gamma RT} = 0.6\sqrt{1.4(287)298} = 207.6177 \text{ m/s}$$

$$\dot{m} = \rho AV = (0.5846)(0.0050 \text{ m}^2)(207.6177 \text{ m/s}) = 0.6069 \text{ kg/s}$$

(b) For $M = 0.8, A/A^* = 1.0382$

For $M = 0.6, A/A^* = 1.1882$

$$(\text{\% reduction in area to reach Mach 0.8}) = \frac{1.1882 - 1.0382}{1.1882}100 = 12.62\%$$

$$(\text{\% reduction in area to reach Mach 1.0}) = \frac{1.1882 - 1}{1.1882}100 = 15.84\%$$

(c) Flow would be reduced.

Problem 6. – A converging nozzle with an exit area of 1.0 cm^2 is supplied from an oxygen reservoir in which the pressure is 500 kPa and the temperature is 1200 K. Calculate the mass flow rate of oxygen for back pressures of 0, 100, 200, 300, and 400 kPa. Assume that γ = 1.3.

For γ = 1.3, the critical pressure ratio is: $\frac{p^*}{p_o} = 0.5457$. So, the back pressure is

$$p_b = \left(\frac{p^*}{p_o}\right) p_o = 0.5457(500) = 272.8500\,\text{kPa}\,,$$

Thus, the nozzle is choked for back-pressures below 272.85 kPa, i.e., for 0, 100, and 200 kPa. For these back pressures, p_e = 272.8 kPa and

$$T_e = \left(\frac{T_e}{T_o}\right) T_o = 0.8696(1200\,\text{K}) = 1043.5200\,\text{K}$$

$$V_e = M_e\sqrt{\gamma R T_e} = \sqrt{1.3(259.8)1043.52} = 593.6653\,\text{m/s}$$

$$\rho_e = \frac{p_e}{RT_e} = \frac{272.85\,\text{kN/m}^2}{(0.2598\ \text{kJ/kg}\cdot\text{K})(1043.52\,\text{K})} = 1.0064\,\text{kg}/\text{m}^3$$

$$\dot{m} = \rho_e A_e V_e = (1.0064)(1\times 10^{-4})(593.6653) = 0.05975\,\text{kg/s}$$

For $p_b = p_e$ =300 kPa; thus, $\frac{p_e}{p_o} = \frac{300}{500} = 0.6$, from which we find $M_e = 0.9133$

$$T_e = \left(\frac{T_e}{T_o}\right) T_o = 0.8888(1200\,\text{K}) = 1066.5600\,\text{K}$$

$$V_e = M_e\sqrt{\gamma R T_e} = 0.9133\sqrt{1.3(259.8)1066.56} = 548.1474\,\text{m/s}$$

$$\dot{m} = \left(\frac{p_e}{RT_e}\right)(1\times 10^{-4}\,\text{m}^2)(546.5\,\text{m/s}) = \frac{300}{0.2598(1066.56)}\frac{\text{kg}}{\text{m}^3}(1\times 10^{-4}\,\text{m}^2)(548.1474\,\text{m/s})$$

$$= 0.05935\ \text{kg/s}$$

For $p_b = p_e$ = 400 kPa, $\frac{p_e}{p_o} = 0.8,\ M_e = 0.5935$

$$T_e = 0.9498(1200) = 1139.7600\,\text{K}$$

$$\dot{m} = \left(\frac{400}{0.2598 \times 1139.76}\right)\left(10^{-4}\right)\left[0.5935\sqrt{1.3(259.8)1139.76}\right] = 0.04974\,\mathrm{kg/s}$$

***Problem** 7*. – Compressed air is discharged through the converging nozzle as shown in Figure P3.7. The tank pressure is 500 kPa, and local atmospheric pressure is 101 kPa. The inlet area of the nozzle is 100 cm^2; the exit area is 34 cm^2. Find the force of the air on the nozzle, assuming the air to behave as a perfect gas with constant γ = 1.4. Take the temperature in the tank to be 300 K.

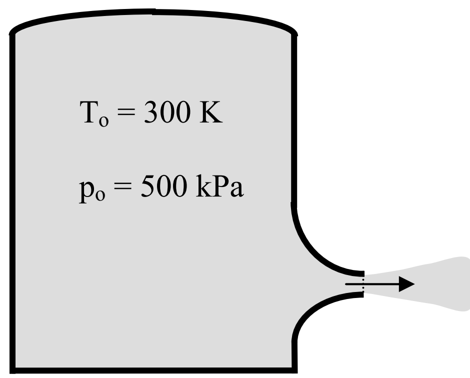


Figure P3.7

Assume the nozzle is choked. Accordingly, p_e = 0.5283 (500 kPa) = 264.15 kPa. Since this pressure exceeds the back pressure, the assumption is valid.

$$M_e = 1.0$$

$$T_e = 0.8333(300) = 249.9900\,\mathrm{K}$$

$$V_e = M_e\sqrt{\gamma R T_e} = \sqrt{1.4(287)249.99} = 316.9321\,\mathrm{m/s}$$

At the nozzle inlet, $\dfrac{A_i}{A^*} = \dfrac{100}{34} = 2.9412$, from which we find $M_i = 0.2038$

$$\frac{T_i}{T_o} = 0.9938, \text{ so } T_i = 0.9938(300) = 298.1400\,\mathrm{K}$$

$$\frac{p_i}{p_o} = 0.9735, \; p_i = 0.9735(500) = 486.7500 \text{ kPa}$$

$$V_i = 0.2038\sqrt{1.4(287)298.14} = 70.5374\,\mathrm{m/s}$$

$$\dot{m}_e = \frac{264.15}{0.287(249.99)}(0.0034)(316.9321) = 3.9673\,\text{kg/s}$$

$$p_i A_i + F_T - p_e A_e - p_{atm}(A_i - A_e) = \dot{m}(V_e - V_i)$$

$$\therefore\ F_T = p_e A_e - p_i A_i + p_{atm}(A_i - A_e) + \dot{m}(V_e - V_i)$$

$$\begin{aligned} F_T = &\left(264.15\,\text{kN/m}^2\right)\left(34\times10^{-4}\,\text{m}^2\right) - \left(486.75\,\text{kN/m}^2\right)\left(100\times10^{-4}\,\text{m}^2\right) \\ &+ \left(101.0\,\text{kN/m}^2\right)(100-34)10^{-4} \\ &+ \frac{(3.9673\,\text{kg/s})(316.9321 - 70.5374\,\text{m/s})}{1000\,\text{N/kN}} \end{aligned}$$

$$F_T = 0.8981 - 4.8675 + 0.6666 + 0.9775 = -2.3253\,\text{kN}$$

The force of the fluid on the nozzle (equal but opposite) is 2.3253 kN to the right.

Problem 8. – A converging nozzle has an exit area of 56 cm. Nitrogen stored in a reservoir is to be discharged through the nozzle to an ambient pressure of 100 kPa. Determine the flow rate through the nozzle for reservoir pressures of 120 kPa, 140 kPa, 200 kPa, and 1 MPa. Assume isentropic nozzle flow. In each case, determine the increase in mass flow to be gained by reducing the back pressure from 100 to 0 kPa. Reservoir temperature is 298 K.

For N_2, γ = 1.40. The nozzle is choked for

$$p_o = \frac{p_b}{(p^*/p_o)} = \frac{100}{0.5283} = 189.2864\ \text{kPa}$$

Case 1. p_o = 120 kPa and p_b = 100 kPa

$$\frac{p_e}{p_o} = \frac{100}{120} = 0.8333,\quad M_e = 0.5171,\quad \frac{T_e}{T_o} = 0.9492$$

$$T_e = 0.9492(298) = 282.8616\,\text{K}$$

$$\rho_e = \frac{p_e}{RT_e} = \frac{100\ \text{kN/m}^2}{(0.2968\ \text{kJ/kg}\cdot\text{k})282.8616\,\text{K}} = 1.1911\ \text{kg/m}^3$$

$$V_e = M_e\sqrt{\gamma R T_e} = 0.5171\sqrt{1.4(296.8)282.8616} = 177.2791\,\text{m/s}$$

$$\dot{m} = \rho_e A_e V_e = (1.1911)(56 \times 10^{-4})177.2791 = 1.1825 \text{ kg/s}$$

Case 2. p_o = 140 kPa and p_b = 100 kPa

$$\frac{p_e}{p_o} = \frac{100}{140} = 0.7143, \quad M_e = 0.7103, \quad T_e = 0.9083(298) = 270.6734 \text{ K}$$

$$\rho_e = \frac{100}{0.2968(270.6734)} = 1.2448 \text{ kg/m}^3$$

$$V_e = 0.7103\sqrt{1.4(296.8)270.6734} = 238.2103 \text{ m/s}$$

$$\dot{m} = 1.2448(56 \times 10^{-4})238.2103 = 1.6605 \text{ kg/s}$$

Case 3. p_o = 200 kPa and p_b = 100 kPa

Since p_o is above the critical reservoir pressure the nozzle is choked, therefore M_e = 1.0

$$p_e = 0.5283(200) = 105.6600 \text{ kPa}$$

$$T_e = 0.8333(298) = 248.3234 \text{ K}$$

$$\rho_e = \frac{105.66}{0.2968(248.3234)} = 1.4336 \text{ kg/m}^3$$

$$V_e = 1.0\sqrt{1.4(296.8)248.3234} = 321.2216 \text{ m/s}$$

$$\dot{m} = (1.4336)(56 \times 10^{-4})(321.2216) = 2.5788 \text{ kg/s}$$

Case 4. p_o = 1 MPa = 1000 kPa and p_b = 100 kPa

$$\dot{m} = 2.5788\left(\frac{1000}{200}\right) = 12.8941 \text{ kg/s}$$

Case 5. p_o = 120 kPa and p_b = 0 kPa

For **Case 1**, lowering back pressure to 0 kPa will change the flow and the nozzle will now be choked. Therefore,

M_e = 1.0,

V_e = 321.2216 m/s

$$\rho_e = \frac{0.5283(120)}{0.2968(248.3234)} = 0.8602\ \text{kg/m}^3$$

$$\dot{m} = (0.8602)\left(56\times10^{-4}\right)(321.2216) = 1.5473\ \text{kg/s}$$

Case 6. p_o = 140 kPa and p_b = 0 kPa

The nozzle is choked, so $M_e = 1$

$$\dot{m} = \frac{140}{120}(1.5473) = 1.8052\ \text{kg/s}$$

Case 7.

$$\dot{m} = \frac{200}{120}(1.5473) = 2.5788\ \text{kg/s}$$

Case 8.

$$\dot{m} = \frac{1000}{120}(1.5473) = 12.8941\ \text{kg/s}$$

Problem 9. – Pressurized liquid water flows from a large reservoir through a converging nozzle. Assuming isentropic nozzle flow with a negligible inlet velocity and a back pressure of 101 kPa, calculate the reservoir pressure necessary to choke the nozzle. Assume that the isothermal compressibility of water is constant at $5 \times 10^{-7}\ (\text{kPa})^{-1}$ and equal to the isentropic compressibility. Exit density of the water is 1000 kg/m^3.

$$\int \frac{dp}{\rho} + \frac{V^2}{2} = c$$

$$k_T \approx k_s = \frac{1}{\rho}\frac{\partial\rho}{\partial p} = \frac{1}{\rho}\frac{d\rho}{dp}$$

$$\frac{1}{k_T}\int_1^2 \frac{d\rho}{\rho^2} + \frac{V_2^2}{2} = 0$$

$$\frac{1}{k_T}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{V_2^2}{2} = 0$$

$$V_2 = a_2 = \sqrt{\frac{1}{\rho_2 k_s}} = \sqrt{\frac{1}{(1000\ \mathrm{kg/m^3})5\times10^{-7}(\mathrm{kPa})^{-1}}} = 1414.2136\ \mathrm{m/s}$$

$$\frac{1}{\rho_1} - \frac{1}{\rho_2} = \left(-\frac{1414.2136^2}{2}\frac{\mathrm{m^2}}{\mathrm{s^2}}\right)\left(5\times10^{-7}\right)\mathrm{kPa}^{-1} = -0.0005\ \mathrm{m^3/kg}$$

$$\frac{1}{\rho_1} = \frac{1}{1000\ \mathrm{kg/m^3}} - 0.0005\ \mathrm{m^3/kg}$$

$$\rho_1 = 2000.0\ \mathrm{kg/m^3}$$

$$\int_1^2 dp = \frac{1}{k_T}\int_1^2 \frac{d\rho}{\rho}$$

$$p_2 - p_1 = \frac{1}{k_T}\ell n\left(\frac{\rho_2}{\rho_1}\right) = \frac{1}{5\times10^{-7}(\mathrm{kPa})^{-1}}\ln\left(\frac{1000}{2000}\right) = -1.3863\times10^6\ \mathrm{kPa}$$

or $$p_r = 101 + 1.3863\times10^6 = 1.3864\times10^6\ \mathrm{kPa}$$

Problem 10. – Calculate the stagnation temperature in an airstream traveling at Mach 5 with a static temperature of 273 K (see Figure P3.10). An insulated flat plate is inserted into this flow, aligned parallel with the flow direction, with a boundary layer building up along the plate. Since the absolute velocity at the plate surface is zero, would you expect the plate temperature to reach the free stream stagnation temperature? Explain.

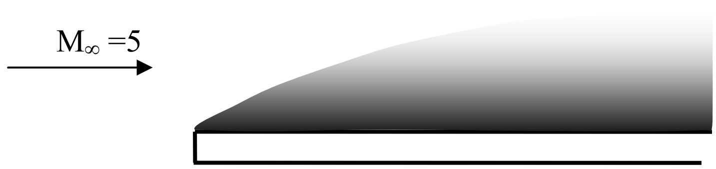
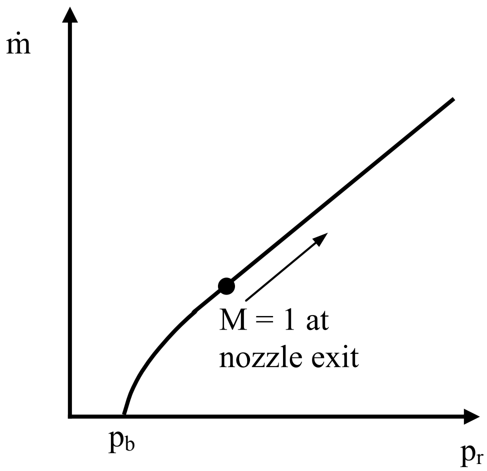


Figure P3.10

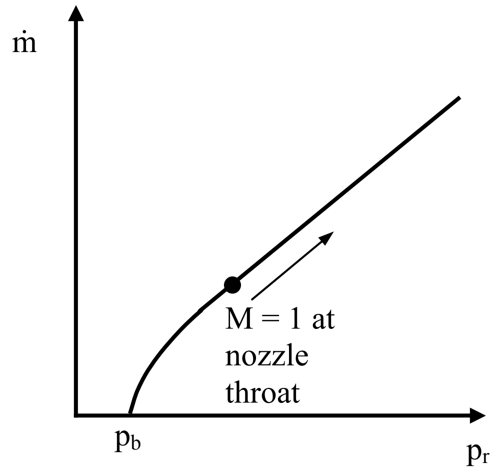
$$T_o = \frac{273}{0.1667} = 1637.7\ \mathrm{K}$$

No. In general the reduction to zero speed is not an adiabatic process. However, it could be if viscous heating counteracts heat conduction back through the boundary layer.

Problem 11. – A gas stored in a large reservoir is discharged through a converging nozzle. For a constant back pressure, sketch a plot of mass flow rate versus reservoir pressure. Repeat for a converging-diverging nozzle.



Converging Nozzle



C-D Nozzle

Problem 12. – A converging-diverging nozzle is designed to operate isentropically with air at an exit Mach number of 1.75. For a constant chamber pressure and temperature of 5 MPa and 200°C, respectively, calculate the following:

(a) Maximum back pressure to choke nozzle
(b) Flow rate in kilograms per second for a back pressure of 101 kPa
(c) Flow rate for a back pressure of 1 MPa Nozzle exit area is 0.12 m^2.

(a) For M = 1.75, $\dfrac{A}{A^*} = 1.386\,5$

For $\dfrac{A}{A^*} = 1.3865, \quad M = 0.4770, \quad \dfrac{p}{p_o} = 0.8558$

Maximum back pressure to choke nozzle = 5(0.8558) = 4.2790 MPa

(b) p_b = 101 kPa, nozzle choked

$$A_{throat} = \frac{0.12\text{m}^2}{1.3865} = 0.086549\ \text{m}^2$$

$$p_{throat} = 5\ \text{MPa}(0.5283) = 2.6415\ \text{MPa}$$

$$T_{throat} = (200 + 273)0.8333 = 394.1509\ \text{K}$$

$$V_{throat} = \sqrt{1.4(287)394.1509} = 397.9571 \text{ m/s}$$

$$\rho_{throat} = \frac{2641.5 \text{ kPa}}{0.287 \frac{\text{kJ}}{\text{kg} \cdot \text{K}}(394.1509 \text{ K})} = 23.3510 \text{ kg/m}^3$$

$$\dot{m} = \rho_{throat} A_{throat} V_{throat} = (23.3510)(0.08655)(397.9571) = 804.2829 \text{ kg/s}$$

(c) $\dot{m} = 804.2829 \text{ kg/s}$

Problem 13. – A supersonic flow is allowed to expand indefinitely in a diverging channel. Does the flow velocity approach a finite limit, or does it continue to increase indefinitely? Assume a perfect gas with constant specific heats.

For adiabatic flow, $c_p T_o = c_p T + \frac{V^2}{2}$. However, T cannot be less than 0 K (second law)

So,

$$V_{max} = \sqrt{2 c_p T_o} \quad \text{and} \quad V_{max} \text{ is finite}$$

Problem 14. – A converging-diverging frictionless nozzle is used to accelerate an airstream emanating from a large chamber. The nozzle has an exit area of 30 cm^2 and a throat area of 15 cm^2. If the ambient pressure surrounding the nozzle is 101 kPa and the chamber temperature is 500 K, calculate the following:

(a) Minimum chamber pressure to choke the nozzle
(b) Mass flow rate for a chamber pressure of 400 kPa
(c) Mass flow rate for a chamber pressure of 200 kPa

(a) $\frac{A_{exit}}{A_{throat}} = 2.0$

For $\frac{A}{A^*} = 2.0$, $M = 0.3059$, $\frac{p}{p_o} = 0.9372$

$$\text{Minimum chamber pressure to choke} = \frac{101}{0.9372} = 107.7678 \text{ kPa}$$

(b) Nozzle choked for $p_c = p_o = 400$ kPa

$$p_{throat} = 0.5283(400) = 211.3200 \text{ kPa}$$

$$T_{throat} = 0.8333(500) = 416.6500\ \text{K}$$

$$\dot{m} = \rho AV = \frac{211.32\ \text{kPa}}{(0.287)\frac{\text{kJ}}{\text{kg}\cdot\text{K}}(416.65\ \text{K})}\left(15\times10^{-4}\ \text{m}^2\right)\sqrt{1.4(287)(416.65)}$$

$$= \left(1.7672\ \text{kg/m}^3\right)\left(15\times10^{-4}\ \text{m}^2\right)409.1576\ \text{m/s}$$

$$= 1.0846\ \text{kg/s}$$

(c) $$\dot{m} = 1.0846\left(\frac{200}{400}\right) = 0.5423\ \text{kg/s}$$

Problem 15. – Sketch p versus x for the case shown in Figure P3.15.

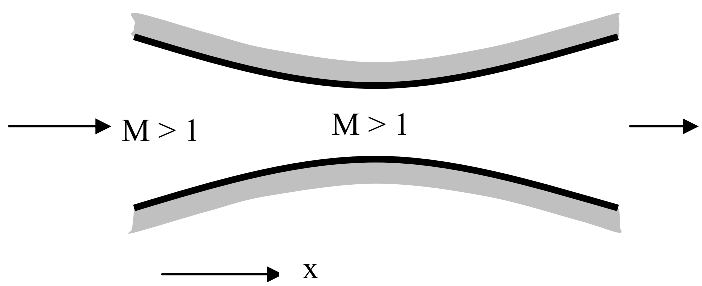
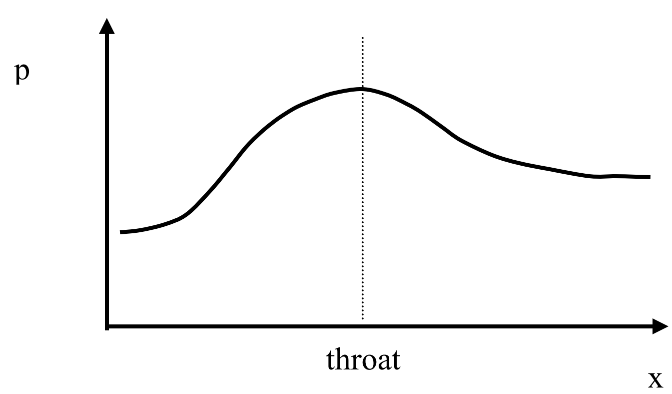


Figure P3.15



Problem 16. – Steam is to be expanded to Mach 2.0 in a converging-diverging nozzle from an inlet velocity of 100 m/s. The inlet area is 50 cm^2; inlet static temperature is 500 K. Assuming isentropic flow, determine the throat and exit areas required. Assume the steam to behave as a perfect gas with constant $\gamma = 1.3$.

$$M_i = \frac{100}{\sqrt{1.3(461.5)500}} = \frac{100}{547.6997} = 0.1826$$

$$\frac{A}{A^*} = 3.2669, \quad A_{throat} = \frac{50}{3.2669} = 15.3050\,\text{cm}^2$$

$$\text{For } M = 2.0, \quad \frac{A}{A^*} = \frac{A_e}{A_t} = 1.7732 \quad \text{so } A_{exit} = 27.1389\,\text{cm}^2$$

Problem 17. – Write a computer program that will yield values of T/T_o, p/p_o, and A/A^* for isentropic flow of a perfect gas with constant $\gamma = 1.27$. Use Mach number increments of 0.05 over the range M = 0 to M = 2.0.

$$b = \frac{\gamma-1}{\gamma+1}, \quad \frac{T}{T_o} = \frac{1-b}{1+b(M^2-1)}, \quad \frac{p}{p_o} = \left(\frac{T}{T_o}\right)^{\frac{b+1}{2b}}, \quad \frac{A}{A^*} = \frac{\left[1+b\left(M^2-1\right)\right]^{\frac{1}{2b}}}{M}$$

M	T/T_o	p/p_o	A/A*
0.00	1.00000	1.00000	infinite
0.05	0.99966	0.99841	11.76142
0.10	0.99865	0.99367	5.90577
0.15	0.99697	0.98584	3.96515
0.20	0.99463	0.97499	3.00342
0.25	0.99163	0.96125	2.43340
0.30	0.98800	0.94478	2.05940
0.35	0.98373	0.92575	1.79759
0.40	0.97886	0.90437	1.60608
0.45	0.97339	0.88086	1.46164
0.50	0.96735	0.85545	1.35034
0.55	0.96076	0.82839	1.26335
0.60	0.95365	0.79994	1.19481
0.65	0.94604	0.77035	1.14069
0.70	0.93795	0.73986	1.09813
0.75	0.92942	0.70873	1.06506
0.80	0.92047	0.67720	1.03995
0.85	0.91113	0.64547	1.02166
0.90	0.90143	0.61378	1.00931

0.95	0.89139	0.58230	1.00226
1.00	0.88106	0.55121	1.00000
1.05	0.87045	0.52067	1.00215
1.10	0.85959	0.49081	1.00844
1.15	0.84851	0.46177	1.01864
1.20	0.83724	0.43362	1.03264
1.25	0.82581	0.40646	1.05032
1.30	0.81423	0.38035	1.07166
1.35	0.80254	0.35534	1.09664
1.40	0.79076	0.33147	1.12530
1.45	0.77891	0.30875	1.15768
1.50	0.76702	0.28718	1.19389
1.55	0.75509	0.26678	1.23404
1.60	0.74316	0.24752	1.27826
1.65	0.73124	0.22939	1.32672
1.70	0.71935	0.21236	1.37960
1.75	0.70750	0.19640	1.43712
1.80	0.69570	0.18147	1.49952
1.85	0.68398	0.16753	1.56703
1.90	0.67234	0.15453	1.63996
1.95	0.66079	0.14244	1.71860
2.00	0.64935	0.13121	1.80329
M	**T/T_o**	**p/p_o**	**A/A***

Problem 18. – A gas is known to have a molecular mass of 18, with c_p = 2.0 kJ/kg · K. The gas is expanded from negligible initial velocity through a converging-diverging nozzle with an area ratio of 5.0. Assuming an isentropic expansion in the nozzle with initial stagnation pressure and temperature 1 MPa and 1000 K, respectively, determine the exit nozzle velocity.

$$R = \frac{8314.3\ \text{J/kg} - \text{mole} \cdot \text{K}}{18\ \text{kg/kg} - \text{mole}} = 461.9056\ \text{J/kg} \cdot \text{K}$$

$$c_p = \frac{R\gamma}{\gamma - 1} = 2.0\ \text{kJ/kg} \cdot \text{K}$$

$$\frac{\gamma}{\gamma - 1} = \frac{2.0}{0.4619056} = 4.3299$$

$$\therefore\ \gamma = 1.300$$

$$\frac{A}{A^*} = 5.0, \quad M_e = 2.9723, \quad T_e = 1000(0.4301) = 430.1000\ \text{K}$$

$$V_e = 2.9723\sqrt{1.3(461.9056)430.1} = 1510.5171\,\text{m/s}$$

Problem 19. – A jet plane is flying at 10 km with a cabin pressure of 101 kPa and a cabin temperature of 20°C. Suddenly a bullet is fired inside the cabin and pierces the fuselage; the resultant hole is 2 cm in diameter. Assume that the temperature within the cabin remains constant and that the flow through the hole behaves as that through a converging nozzle with an exit diameter of 2.0 cm. Take the cabin volume to be 100 m^3. Calculate the time for the cabin pressure to decrease to one-half the initial value. At 10 km, p = 26.5 kPa and T = 223.3 K.

Because the back pressure to cabin pressure is 26.5/101 = 0.2624, which is less than 0.5283 the critical pressure ratio at γ = 1.4, the flow is choked and Me = 1. Hence, the mass flow rate is

$$\dot{m} = \rho AV = \frac{p}{RT}AM\sqrt{\gamma RT} = \frac{(0.5283)p_c}{287(0.8333)(293)}\left(\frac{\pi}{4}0.02^2\right)(1)\sqrt{1.4(287)(0.8333)(293)}$$

$$= 7.4186\times10^{-7}\,p_c$$

In the cabin,

$$p_c\forall = mRT$$

$$\therefore \frac{dp_c}{dt} = \frac{RT}{\forall}\frac{dm}{dt} = -\frac{RT}{\forall}\dot{m}$$

$$\frac{dp_c}{p_c} = -\frac{RT}{\forall}\left(7.4186\times10^{-7}\right)dt$$

Integration produces,

$$\ln\frac{p_{c\,final}}{p_{c\,initial}} = -\frac{RT}{\forall}\left(7.4186\times10^{-7}\right)t$$

$$\ln 2 = \frac{(287)(293)}{100}\left(7.4186\times10^{-7}\right)t$$

$$t = 1111.1096\text{ s} = 0.3086\text{ h}$$

Problem 20. – A rocket nozzle is designed to operate isentropically at 20 km with a chamber pressure of 2.0 MPa and chamber temperature of 3000 K. If the products of combustion are assumed to behave as a perfect gas with constant specific heats ($\gamma = 1.3$ and MM = 20), determine the design thrust for a nozzle throat area of 0.25 m^2.

At 20 km, p = 5.53 kPa

$$\frac{p_b}{p_r} = \frac{5.53}{2000} = 0.002765 = \frac{p_e}{p_o}$$

$$M_e = 4.3923, \quad T_e = \left(\frac{T}{T_o}\right)_e T_o = 0.2568(3000) = 770.4000\ \mathrm{K}$$

At design

$$\text{Thrust} = mV_e + (p_e)A_e$$

$$V_e = 4.3923\sqrt{1.3\left(\frac{8314.3}{20}\right)770.4} = 2834.1293\ \mathrm{m/s}$$

Now at the throat $M_t = 1$, so $(p/p_o)_t = 0.5457$ and $(T/T_o)_t = 0.8696$.

$$\dot{m}_t = \left[\frac{(0.5457)2000\ \mathrm{kN/m^2}}{\left(\frac{8.3143}{20}\frac{\mathrm{kNm}}{\mathrm{kg\cdot K}}\right)(0.8696)(3000)\mathrm{K}}\right]\left(0.25\mathrm{m}^2\right)(1)\sqrt{1.3\left(\frac{8314.3}{20}\right)(0.8696)(3000)}\ \mathrm{m/s}$$

$$= \left(1.0063\ \mathrm{kg/m^3}\right)\left(0.25\mathrm{m}^2\right)\left(1187.3805\ \mathrm{m/s}\right)$$

$$= 298.7290\ \mathrm{kg/s}$$

$$\begin{aligned}\text{Thrust} &= \left[(298.7290)(2834.1293) + (5530)(0.25)\right]/(1{,}000{,}000)\\ &= 0.8466 + 0.0014\\ &= 0.8480\mathrm{MN}\end{aligned}$$

Problem 21. – A converging nozzle has a rectangular cross section of a constant width of 10 cm. For ease of manufacture, the sidewalls of the nozzle are straight, making an angle of 10° with the horizontal, as shown in Figure P3.21. Determine and plot the variation of M, T, and p with x, taking $M_1 = 0.4$, $P_{o1} = 200$ kPa, and $T_{o1} = 350$ K. Assume the working fluid to be air, which behaves as a perfect gas with constant specific heats ($\gamma = 1.4$), and that the flow is isentropic.

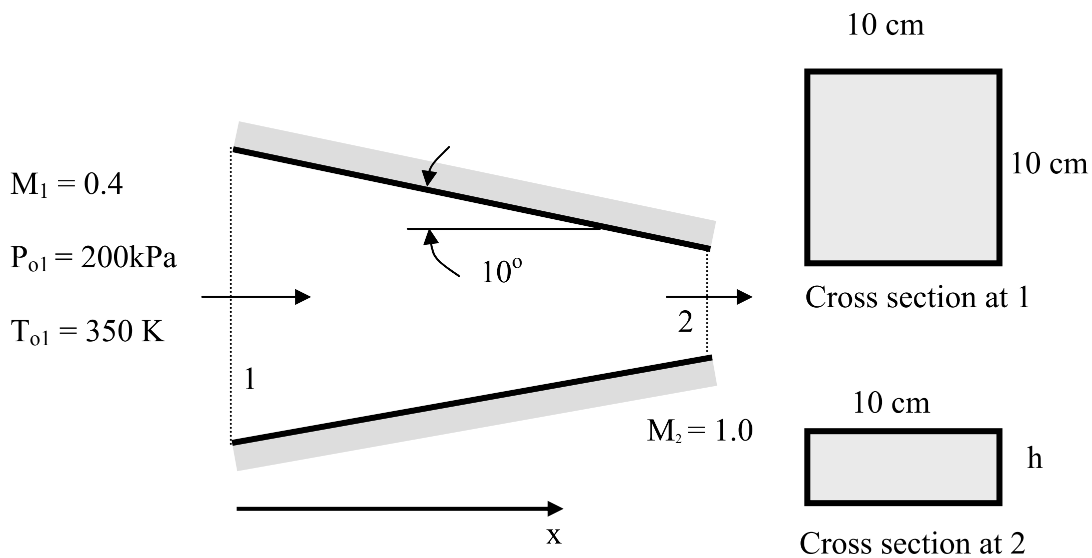


Figure P3.21

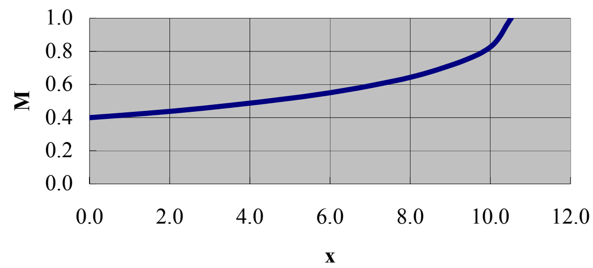
. $A_2 = A^*$

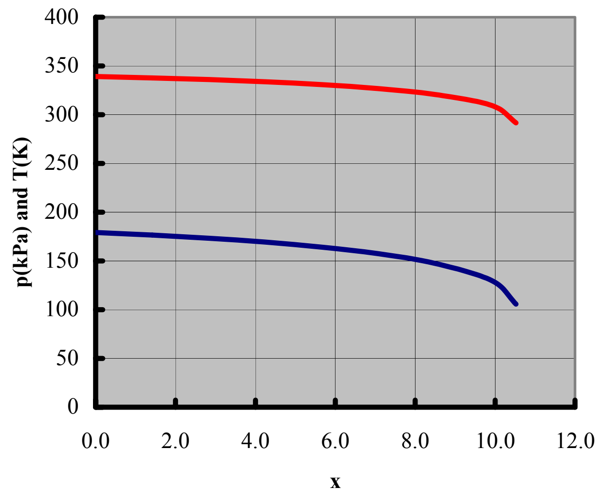
$$A_1 = 100\,\text{cm}^2,\; M_1 = 0.4,\; \frac{A_1}{A^*} = 1.5901$$

$$A_2 = \frac{100}{1.5901} = 62.89\,\text{cm}^2, \quad \text{so } h = 6.289\,\text{cm}$$

$$x = \frac{h_1 - h}{2\tan(10)}$$

h (cm)	x (cm)	A/A*	M	p (kPa)	T (K)
10.00	0	1.5901	0.400	179.1	339.2
9.50	1.418	1.5106	0.426	176.5	337.7
9.00	2.836	1.4311	0.457	173.3	336.0
8.50	4.253	1.3516	0.494	169.3	333.7
8.00	5.671	1.2721	0.539	164.1	330.8
7.50	7.089	1.1926	0.596	157.3	326.8
7.00	8.507	1.1131	0.676	147.3	320.7
6.50	9.925	1.0336	0.812	129.7	309.3
6.29	10.523	1.0000	1.000	105.7	291.7





Problem 22. – A spherical tank contains compressed air at 500 kPa; the volume of the tank is 20 m^3. A 5-cm burst diaphragm in the side of the tank ruptures, causing air to escape from the tank. Find the time required for the tank pressure to drop to 200 kPa. Assume the temperature of the air in the tank remains constant at 280 K, the ambient pressure is 101 kPa and that the airflow through the opening can be treated as isentropic flow through a converging nozzle with a 5-cm exit diameter.

For $p_{tank} = 200\,\text{kPa}, \ \dfrac{p_b}{p_o} = \dfrac{101}{200} = 0.505\,(< 0.5283 \text{ so choked})$

$$p_e = 0.5283\,p_o, \quad T_e = 0.8333(280) = 233.3240\,\text{K}$$

$$V_e = \sqrt{\gamma R T_e} = \sqrt{1.4(287.0)233.3240} = 306.1855\,\text{m/s}$$

$$\dot{m} = \frac{0.5283\,p_o}{0.287(233.3240)}\left(\frac{\pi}{4}0.05^2\right)306.1855$$

$$= 0.004743\,p_o\ \text{kg/s} \quad \text{with} \quad p_o \text{ in kPa}$$

In the tank,

$$p_o \forall = mRT$$

$$\frac{dp_o}{dt} = \frac{RT}{\forall}\frac{dm}{dt} = -\frac{RT}{\forall}\dot{m} = -\frac{RT}{\forall}\left(0.004743\,p_o\right)$$

$$\frac{dp_o}{p_o} = \frac{-0.287(270)}{20}(0.004743)\,dt$$

$$\ln\frac{200}{500} = -0.01838\,t$$

$$t = \frac{\ln(0.4)}{-0.01838} = 49.8526\,s$$

Problem 23. – A converging-diverging nozzle has an area ratio of 3.3 to 1. The nozzle is supplied from a tank containing a gas at 100 kPa and 270 K (see Figure P3.23). Determine the maximum mass flow possible through the nozzle and the range of back pressures over which the mass flow can be attained assuming the gas is (a) helium (γ = 1.67, R = 2.077 kJ/kg·K) and (b) hydrogen (γ = 1.4, R = 4.124 kJ/kg·K).

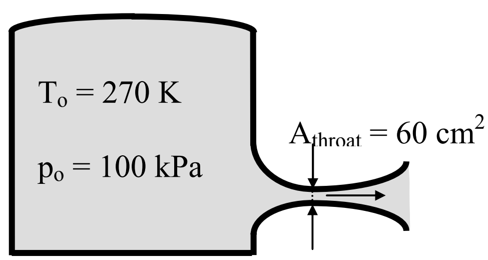


Figure P3.23

(a) Helium: $\gamma = 1.67,\ \frac{A_e}{A^*} = 3.3$

$$M_e = 0.1739,\ 3.1494$$

Maximum p_b to choke nozzle: at M_e = 0.1739, $\left(\frac{p}{p_o}\right)_e = 0.9752$

Maximum p_b to choke nozzle = 97.52 kPa

Nozzle choked for $p_b \le 97.52$ kPa

$$\dot{m}_{max} = \frac{p_{throat}}{RT_{throat}} AM_{throat}\sqrt{\gamma RT_{throat}}$$

$$= \frac{(0.4867)100}{2.077(0.7491)270}\left(60\times10^{-4}\right)(1)\sqrt{1.67(2077)(0.7491)270}$$

$$\dot{m}_{max} = 0.5822 \text{ kg/s}$$

(b) Hydrogen: $\gamma = 1.40, \ \dfrac{A_e}{A^*} = 3.3$

$$M_e = 0.1787, \quad \left(\frac{p}{p_o}\right)_e = 0.9780$$

Nozzle choked for all $p_b \leq 97.8$ kPa

$$\dot{m}_{max} = \frac{(0.5283)100}{4.124(0.8333)270}\left(60\times10^{-4}\right)(1)\sqrt{1.4(4124)(0.8333)270}$$

$$= 0.3894 \text{ kg/s}$$

Problem 24. – Superheated steam is stored in a large tank at 6 MPa and 800°C. The steam is exhausted isentropically through a converging-diverging nozzle. Determine the velocity of the steam flow when the steam starts to condense, assuming the steam to behave as a perfect gas with $\gamma = 1.3$.

Solution Using Steam Table Data

At 6 MPa, 800°C: $s_1 = 7.6554 \text{ kJ/kg}\cdot\text{K}$

$$h_1 = u_1 + p_1 v_1$$

$$= 3641.2\frac{\text{kJ}}{\text{kg}} + (6000)(0.08159)\frac{\text{kJ}}{\text{kg}}$$

$$= 4130.7 \text{ kJ/kg}$$

Steam will just condense for $s_2 = s_g = s_1$

At 45 kPa, $s_g = 7.6307$; at 40 kPa, $s_g = 7.6709$

Interpolation gives

$$p_2 = 42\,\text{kPa},\quad T_2 = 77°\text{C},\quad h_2 = 2638.8\,\frac{\text{kJ}}{\text{kg}}$$

$$V_2 = \sqrt{2(h_1 - h_2)} = \sqrt{2(4130.7 - 2638.8)1000} = 1727\ \text{m/s}$$

Solution Assuming Steam is a Perfect Gas

$$\frac{p}{p_o} = \frac{42}{6000} = 0.007,\quad M_e = 3.7794,\quad T_e = 273 + 600 = 873\,\text{K}$$

$$T_2 = 873\ (0.3182)\ = 277.7886\ \text{K}$$

$$V_2 = 3.7794\sqrt{1.3(461.5)277.7886} = 1542.8994\ \text{m/s}$$

Because the second answer assumes that the steam is a perfect gas with constant specific heats, the first answer is more accurate.

Problem 25. – Air is stored in a tank 0.037661m^3 in volume at an initial pressure of 5,760.6 kPa and a temperature of 321.4K. The gas is discharged through a converging nozzle with an exit area of 3.167x10^{-5} m^2. For a back-pressure of 101 kPa, assuming a spatially lumped polytropic process in the tank, i.e., pv^n = constant, and isentropic flow in the nozzle, i.e., pv^γ = constant, compare predicted tank pressures to the measured values contained the following table. Try various values of the polytropic exponent, n, from 1.0 (isothermal) to 1.4 (isentropic). Perform only a Stage I analysis, i.e., the nozzle is choked.

time, sec	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	9.0	11.0	13.0	15.0	17.0	19.0
p_o/p_{o1}	1.000	0.717	0.551	0.448	0.358	0.281	0.241	0.199	0.142	0.104	0.078	0.059	0.044	0.033

Now from the continuity equation

$$\frac{dm}{dt} = -\dot{m}_e = -\rho_e A_e V_e$$

For polytropic expansion within the tank

$$\frac{p_o}{\rho_o^n} = \frac{p_{o1}}{\rho_{o1}^n}$$

So

$$\rho_o = \rho_{o1}\left(\frac{p_o}{p_{o1}}\right)^{\frac{1}{n}}$$

And for isentropic expansion in the nozzle

$$\frac{p_o}{\rho_o^\gamma} = \frac{p_e}{\rho_e^\gamma}$$

So

$$\rho_e = \rho_o\left(\frac{p_o}{p_{o1}}\right)^{\frac{1}{\gamma}}$$

For a choked flow: $M_e = 1$, $V_e = a_e = \sqrt{\gamma p_e/\rho_e}$ and

$$\frac{p_o}{p_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\gamma/(\gamma-1)} = \left(\frac{\gamma + 1}{2}\right)^{\gamma/(\gamma-1)}$$

So,

$$p_e = p_o\left(\frac{\gamma + 1}{2}\right)^{\gamma/(1-\gamma)}$$

Therefore,

$$\rho_e = \rho_{o1}\left(\frac{\gamma + 1}{2}\right)^{\frac{1}{1-\gamma}}\left(\frac{p_o}{p_{o1}}\right)^{\frac{1}{n}}$$

Now,

$$\dot{m}_e = \rho_e A_e V_e = \rho_e A_e a_e = \rho_e A_e \sqrt{\gamma \frac{p_e}{\rho_e}} = A_e \sqrt{\gamma p_e \rho_e}$$

$$= \sqrt{\gamma} A_e p_o^{1/2} \rho_{o1} \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{2(1-\gamma)}} \rho_{o1}^{1/2} \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{2(1-\gamma)}} \left(\frac{p_o}{p_{o1}}\right)^{\frac{1}{2n}}$$

Using $a_{o1} = \sqrt{\gamma p_{o1}/\rho_{01}}$ the mass flow rate at the exit can be written as

$$\dot{m}_e = A_e a_{o1} \frac{\rho_{o1}}{p_{o1}^{1/2}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} \left(\frac{1}{p_{o1}}\right)^{\frac{1}{2n}} p_o^{\frac{n+1}{2n}}$$

Now the time rate of change of the mass within the tank is given by

$$\frac{dm}{dt} = \forall \frac{d\rho_o}{dt} = \left[\frac{\rho_{o1}\forall}{n\, p_{o1}^{1/n}}\right] p_o^{\frac{(1-n)}{n}} \frac{dp_o}{dt}$$

Equating this to the exiting flow rate gives

$$\left[\frac{\rho_{o1}\forall}{n\, p_{o1}^{1/n}}\right] p_o^{\frac{(1-n)}{n}} \frac{dp_o}{dt} = -\dot{m}_e = -A_e a_{o1} \frac{\rho_{o1}}{p_{o1}^{1/2}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} \left(\frac{1}{p_{o1}}\right)^{\frac{1}{2n}} p_o^{\frac{n+1}{2n}}$$

or

$$\frac{p_o^{\frac{1-n}{n}}}{p_o^{\frac{n+1}{2n}}} \frac{dp_o}{dt} = p_o^{\left(\frac{1-n}{n}\right)-\left(\frac{n+1}{2n}\right)} \frac{dp_o}{dt} = p_o^{\frac{1-3n}{2n}} \frac{dp_o}{dt} = \frac{-nA_e a_{o1}}{\forall} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_{o1}^{1/n-1/2-1/2n}$$

$$= \frac{-nA_e a_{o1}}{\forall} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_{o1}^{\frac{1-n}{2n}}$$

Integration yields, (note: $\frac{1-3n}{2n} + 1 = \frac{1-n}{2n}$)

$$p_o^{\frac{1-n}{2n}} - p_{o1}^{\frac{1-n}{2n}} = \left(\frac{n-1}{2}\right)\left(\frac{A_e a_{o1}}{\forall}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_{o1}^{\frac{1-n}{2n}}\, t$$

Rearrangement brings

$$\frac{p_o}{p_{o1}} = \left[1 + \left(\frac{n-1}{2}\right)\left(\frac{A_e a_{o1}}{\forall}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} t\right]^{\frac{2n}{1-n}}$$

Note this is not valid for n =1, the isothermal case which must be treated separately. For n = 1

$$\dot{m}_e = A_e a_{o1} \frac{\rho_{o1}}{p_{o1}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_o$$

$$\rho_o = \rho_{o1}\left(\frac{p_o}{p_{o1}}\right)$$

$$\frac{dm}{dt} = \forall \frac{d\rho_o}{dt} = \forall \frac{\rho_{o1}}{p_{o1}} \frac{dp_o}{dt} = -\dot{m}_e = -A_e a_{o1} \frac{\rho_{o1}}{p_{o1}} \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} p_o$$

Canceling, separating variables, integrating and rearranging yields,

$$\frac{p_o}{p_{o1}} = e^{-\frac{A_e a_{o1}}{\forall}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(1-\gamma)}} t}$$

A spreadsheet program was written and run for various n. A table of the results is as follows

n =	1.0	1.1	1.2	1.3	1.4	
t	**p_o/p_{o1}**	**p_o/p_{o1}**	**p_o/p_{o1}**	**p_o/p_{o1}**	**p_o/p_{o1}**	**p_o/p_{o1} (exp)**
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.000
1.0	0.8396	0.8257	0.8122	0.7990	0.7862	0.717
2.0	0.7049	0.6830	0.6620	0.6421	0.6230	0.551
3.0	0.5918	0.5658	0.5415	0.5187	0.4974	0.448
4.0	0.4969	0.4695	0.4443	0.4212	0.3999	0.358
5.0	0.4172	0.3902	0.3658	0.3437	0.3237	0.281
6.0	0.3503	0.3247	0.3021	0.2818	0.2636	0.241
7.0	0.2941	0.2707	0.2502	0.2321	0.2159	0.199
9.0	0.2073	0.1890	0.1731	0.1594	0.1473	0.142
11.0	0.1461	0.1327	0.1211	0.1112	0.1025	0.104
13.0	0.1030	0.0937	0.0856	0.0787	0.0726	0.078
15.0	0.0726	0.0665	0.0611	0.0564	0.0522	0.059
17.0	0.0512	0.0474	0.0440	0.0410	0.0382	0.044
19.0	0.0361	0.0340	0.0320	0.0301	0.0283	0.033

Chapter Four

STATIONARY NORMAL SHOCK WAVES

Problem 1. – A helium flow with a velocity of 2500 m/s and static temperature of 300 K undergoes a normal shock. Determine the helium velocity and the static and stagnation temperatures after the wave. Assume the helium to behave as a perfect gas with constant $\gamma = 5/3$ and R = 2077 J/kg·K.

$$M_1 = \frac{2500}{\sqrt{(5/3)(2077)300}} \frac{2500}{1766.4219} = 1.4153$$

From the normal shock relations

$$\frac{T_2}{T_1} = 1.2646, \quad T_2 = 300(1.2646) = 379.3800 \text{ K}$$

From the isentropic relations

$$\frac{T_1}{T_{o1}} = 0.7140, \quad T_{o2} = T_{o1} = \frac{300}{0.714} = 420.1681 \text{ K}$$

From the normal shock relations

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = 1.7162, \quad V_2 = \frac{2500}{1.7162} = 1456.7067 \text{ m/s}$$

Problem 2. – A normal shock occurs at the inlet to a supersonic diffuser, as shown in Figure P4.2. A_e/A_i is equal to 3.0. Find M_e, p_e, and the loss in stagnation pressure ($p_{oi} - p_{oe}$). Repeat for a shock at the exit. Assume $\gamma = 1.4$.

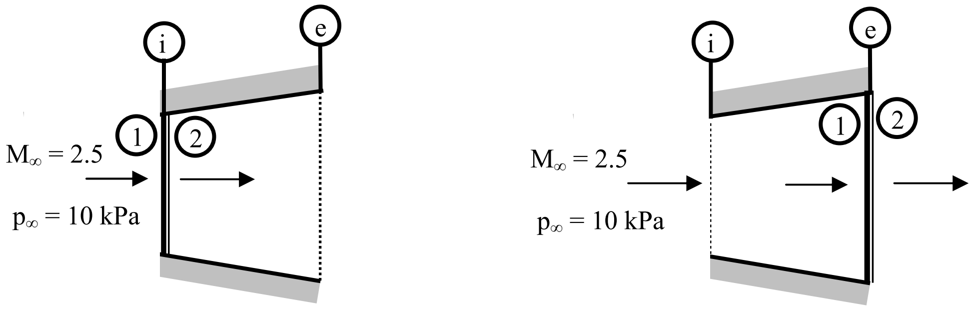


Figure P4.2

Shock at inlet:

$$M_\infty = M_1 = 2.5,\ \frac{p_{o2}}{p_{o1}} = 0.4990 = \frac{A_1*}{A_2*},\ \frac{p_1}{p_{o1}} = 0.05853$$

$$\frac{A_e}{A_e^*} = \frac{A_e}{A_2^*} = \left(\frac{A_e}{A_1}\right)\left(\frac{A_1}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = (3.0)(2.6367)(0.4990) = 3.9471$$

$$M_e = 0.1486$$

$$\frac{p_e}{p_{o2}} = 0.9847,\quad \frac{p_{o2}}{p_{o1}} = 0.4990,$$

$$p_{o1} = \frac{10}{0.05853} = 170.8526 \text{ kPa},\quad p_{o2} = 0.4990(170.8526) = 85.2554 \text{ kPa}$$

$$p_e = \frac{p_e}{p_{o2}} p_{o2} = 0.9847(85.2554) = 83.9510 \text{ kPa}$$

$$p_{oi} - p_{oe} = p_{o1}\left(1 - \frac{p_{o2}}{p_{o1}}\right) = 170.8526(1 - 0.4990) = 85.5972 \text{ kPa}$$

Shock at exit:

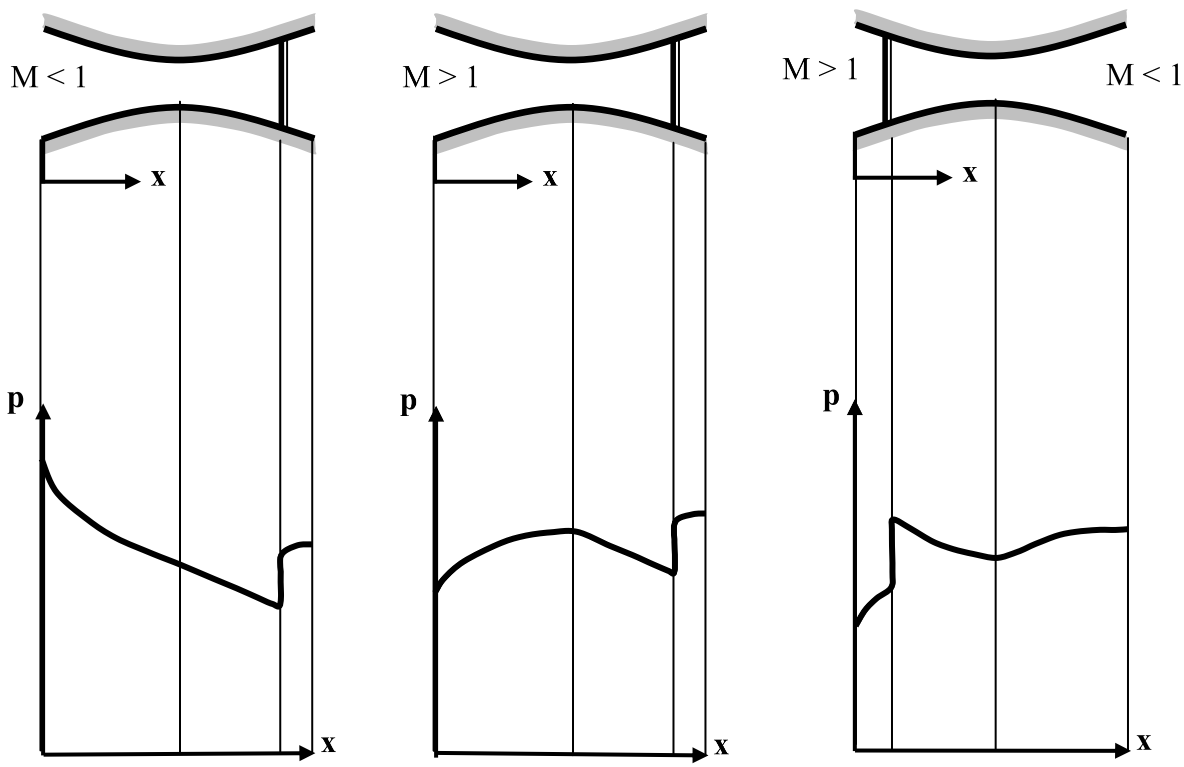
$$\frac{A_1}{A_i} = 3.0,\quad \frac{A_i}{A_1*} = 2.6367,\quad \frac{A_1}{A_1*} = 3(2.6367) = 7.9101$$

$$M_1 = 3.6649,\quad M_2 = 0.4451$$

$$p_2 = \frac{p_2}{p_1}\frac{p_1}{p_{o1}}\frac{p_{o1}}{p_i} p_i = (15.5038)(0.01040)\left(\frac{1}{0.05853}\right)10 = 27.5482 \text{ kPa}$$

$$p_{oi} - p_{ob} = p_{oi}\left(1 - \frac{p_{o2}}{p_{o1}}\right) = 170.8526(1 - 0.1847) = 139.2961\,\text{kPa}$$

Problem 3. – Sketch p versus x for the three cases shown in Figure P4.3. Assume isentropic flow except for flow across the normal shocks.



Problem 4. – Air expands from a storage tank through a converging-diverging nozzle (see Figure P4.4). Under certain conditions it is found that a normal shock exists in the diverging section of the nozzle at an area equal to twice the throat area, with the exit area of the nozzle equal to four times the throat area. Assuming isentropic flow except for shock waves, that the air behaves as a perfect gas with constant γ = 1.4, and that the storage tank pressure and temperature are 200 kPa and 300 K, determine the following:

(a) A* for flow from inlet to shock
(b) A* for flow from shock to exit
(c) Mach number at nozzle exit plane
(d) Stagnation pressure at nozzle exit plane
(e) Exit plane static pressure
(f) Exit plane velocity

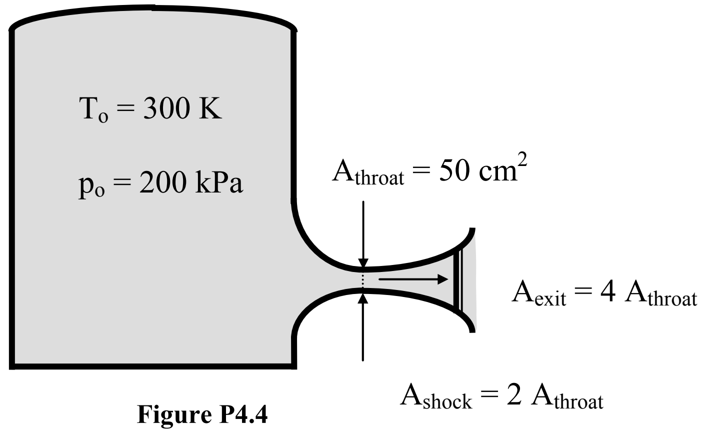


Figure P4.4

(a) 50 cm^2

(b) For shock, $M_1 = 2.20, \dfrac{p_{o2}}{p_{o1}} = \dfrac{A_1^*}{A_2^*} = 0.6281$

$$A^*_{\text{shock to exit}} = A_2^* = \frac{50}{0.6281} = 79.6052 \text{ cm}^2$$

(c) $\dfrac{A_e}{A_2^*} = \dfrac{200}{79.6052} = 2.5124, \quad M_e = 0.2383$

(d) $p_{oe} = p_{o1}\left(\dfrac{p_{o2}}{p_{o1}}\right) = 200(0.6281) = 125.6200 \text{ kPa}$

(e) $p_e = p_{oe}\left(\dfrac{p_e}{p_{oe}}\right) = 125.62(0.9613) = 120.3548 \text{ kPa}$

(f) $T_e = T_o\left(\dfrac{T_e}{T_o}\right) = 300(0.9888) = 296.6400 \text{ K}$

$$V_e = 0.2383\sqrt{1.4(287)296.64} = 82.2704 \text{ m/s}$$

Problem 5. – A supersonic flow at Mach 3.0 and $\gamma = 1.4$ is to be slowed down via a normal shock in a diverging channel. For the conditions shown in Figure P4.5, find p_2/p_1 and p_e/p_i.

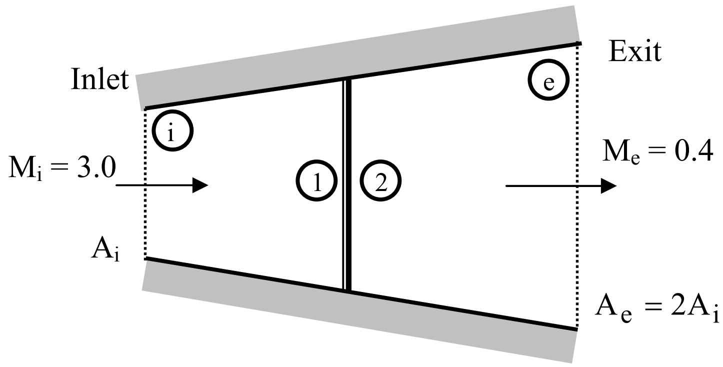


Figure P4.5

From the isentropic Mach number-area relation at the inlet and exit Mach numbers, we have

$$\frac{A_i}{A_i^*} = 4.2346, \quad \frac{A_e}{A_e^*} = 1.5901, \text{ and from the given information } \frac{A_i}{A_e} = 0.50$$

Hence,

$$\frac{A_i^*}{A_e^*} = \frac{A_i^*}{A_i}\frac{A_i}{A_e}\frac{A_e}{A_e^*} = \left(\frac{1}{4.2346}\right)(0.50)(1.5901) = 0.1878 = \frac{p_{oe}}{p_{oi}} = \frac{p_{o2}}{p_{o1}}$$

Now using this ratio of stagnation pressures across the shock, we can find the Mach number on the upstream side of the shock, i.e., M_1, and in turn, determine the pressure ratio across the shock: $M_1 = 3.6455$

$$\frac{p_2}{p_1} = 15.3378$$

$$\frac{p_e}{p_i} = \frac{p_e}{p_{oe}}\frac{p_{oe}}{p_{oi}}\frac{p_{oi}}{p_i} = (0.8956)(0.1878)\left(\frac{1}{0.02722}\right) = 6.1790$$

Problem 6. – A body is reentering the earth's atmosphere at a Mach number of 20. In front of the body is a shock wave, as shown in Figure P4. 7. Opposite the nose of the body, the shock can be seen to be normal to the flow direction. Determine the stagnation pressure and temperature to which the nose is subjected. Assume that the air behaves as a perfect gas (neglect dissociation) with constant $\gamma = 1.4$. The ambient pressure and temperature are equal to 1.0 kPa and 220 K.

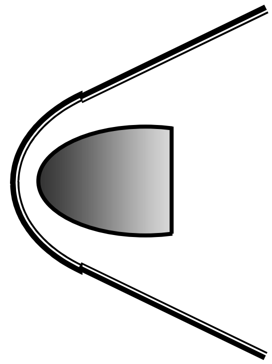


Figure P4.6

$$M_2^2 = \frac{M_1^2 + \dfrac{2}{\gamma - 1}}{\dfrac{2\gamma}{\gamma - 1} M_1^2 - 1} = 0.1447, \quad M_2 = 0.3804$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} = 466.5000, \quad p_2 = 466.5000 \text{ kPa}$$

$$\frac{T_2}{T_1} = \frac{1 + \dfrac{\gamma - 1}{2} M_1^2}{1 + \dfrac{\gamma - 1}{2} M_2^2} = 78.7219, \quad T_2 = 17{,}318.8125 \text{ K}$$

$$p_{o2} = \frac{p_{o2}}{p_2} p_2 = \left(\frac{1}{0.90497} \right) 466.5 = 515.4867 \text{ kPa}$$

$$T_{o2} = \frac{T_{o2}}{T_2} T_2 = \left(\frac{1}{0.9719} \right) 17318.8125 = 17{,}819.5416 \text{ K}$$

***Problem** 7*. – Determine the back pressure necessary for a normal shock to appear at the exit of a converging-diverging nozzle, as shown in Figure P4.7. Assume $\gamma = 1.4$.

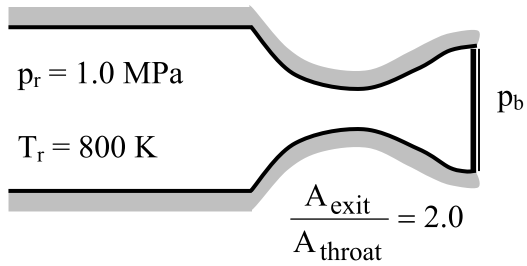


Figure P4.7

From the given area ratio, we use the Newton-Raphson method to determine the supersonic Mach number on the upstream side of the shock. Then we may use the isentropic and shock relations to determine the pressure ratios that enable us to compute the back pressure:

$$M_1 = 2.1972, \ \frac{p_1}{p_{o1}} = .09393, \quad \frac{p_2}{p_1} = 5.4656$$

$$p_b = \frac{p_b}{p_e}\frac{p_e}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{o1}}p_{o1} = (1)(1)(5.4656)(0.09393)1.0 = 0.51338 \text{ MPa} = 513.38 \text{ kPa}$$

Problem 8. – A normal shock is found to occur in the diverging portion of a converging-diverging nozzle at an area equal to 1.1 times the throat area. If the nozzle has a ratio of exit area to throat area of 2.2, determine the percent of decrease in nozzle exit velocity due to the presence of the shock (compared with the exit velocity of a perfectly expanded isentropic supersonic nozzle flow). Assume the flow is expanded from negligible velocity, that the stagnation temperature of the flow is the same for both cases, and that the working fluid is steam, which behaves as a perfect gas with constant $\gamma = 1.3$.

With no shock,

From the given area ratio and because the flow is choked: $A_e/A_t = A_e/A^* = 2.2$, we can determine the exit Mach number using the Newton-Raphson method and find that Me = 2.2201, and therefore, the static to total temperature ratio is 0.5749. Hence,

$$V_e = M_e\sqrt{\gamma RT_e} = 2.2201\sqrt{\gamma RT_o(0.5749)}$$

With shock,

$$\frac{A_s}{A_1^*} = \frac{A_1}{A_1^*} = 1.1, \quad \text{so using Newton - Raphson we find} \quad M_1 = 1.3598$$

$$\frac{p_{o2}}{p_{o1}} = \frac{A_1^*}{A_2^*} = 0.9662$$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_1^*}\frac{A_1^*}{A_2^*} = (2.2)(1)(0.9662) = 2.1256$$

From this area ratio we are able to extract the exit Mach number again using the Newton-Raphson method, therefore, the static to total temperature ratio

$$M_e = 0.2888, \quad \frac{T_e}{T_o} = 0.9876$$

$$V_e = 0.2888\sqrt{\gamma R T_o (0.9876)}$$

$$\% \text{ decrease in } V_e = 100\left(\frac{2.2201\sqrt{\gamma R T_o(0.5749)} - 0.2888\sqrt{\gamma R T_o(0.9876)}}{2.2201\sqrt{\gamma R T_o(0.5749)}}\right)$$

$$= 100\left(1 - \frac{0.2870}{1.6833}\right) = 82.9502\% \text{ decrease}$$

Problem 9. – A flow system consists of two converging-diverging nozzles in series (see Figure P4.9a. If the area ratio (exit to throat) of each nozzle is 3.0 to 1, find the area ratio A_3/A_1 necessary to produce sonic flow at the second throat, with a shock at A_2. Assume isentropic flow except for the normal shock. Find the percent of loss in stagnation pressure for this flow. At another operating condition, a shock appears at A_3 (Figure P4.9b). Find the percent of loss of stagnation pressure for this condition.

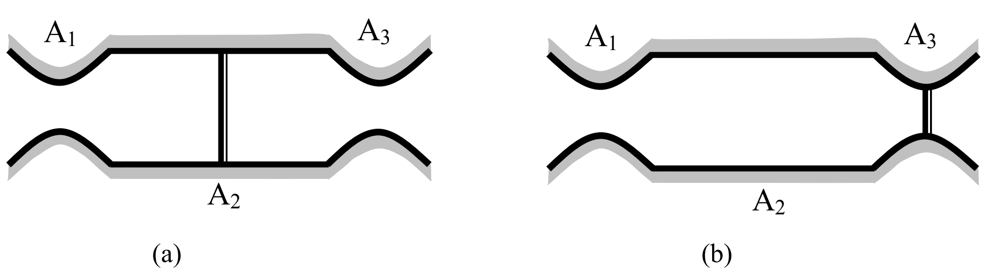


Figure P4.9

(a) *For the shock at A_2*, we may use the given area ratio to determine the Mach number exiting the upstream nozzle and assuming that the Mach number does not change in the constant area section we then have

$$M_1 = 2.6374,\ \frac{p_{o2}}{p_{o1}} = \frac{A_1^*}{A_2^*} = 0.4462,\text{ so }\ \frac{A_2^*}{A_1^*} = 2.2411$$

Since sonic flow exists at both A_1 and A_3, we have, $\dfrac{A_3}{A_1} = \dfrac{A_2^*}{A_1^*} = 2.2411$

$$\%\text{ loss in stagnation pressure} = \left(\frac{p_{o1} - p_{o2}}{p_{o1}}\right)100 = (1 - 0.4462)100 = 55.3800\%$$

(b) *For shock at A_3*, we have from part (a) $A_3/A_1 = A_3/A^* = 2.411$. Using this area ratio, we can find the Mach number on the upstream side of the shock, i.e.,

$$M_1 = 2.3238$$

And so,

$$\frac{p_{o2}}{p_{o1}} = 0.5728$$

or 42.72% loss of stagnation pressure

Problem 10. – For the system shown in Figure P4.10, $M_i = 2.0$, $A_i = 20\text{ cm}^2$, throat area = 15 cm^2, shock area = 22 cm^2, and exit area = 25 cm^2. With the working fluid behaving as a perfect gas with constant $\gamma = 1.3$, find the following:

(a) Throat Mach number
(b) Exit Mach number
(c) Ratio of exit static pressure to static pressure at i

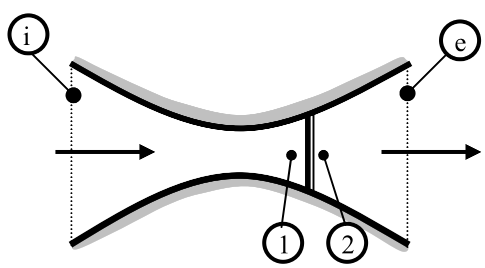


Figure P4.10

(a) Now at $M_i = 2.0$ and for $\gamma = 1.3$, we use the Mach number-area relation to find:

$$\frac{A_i}{A_1^*} = 1.7732\ .$$

Hence,

$$\frac{A_t}{A_1^*} = \frac{A_t}{A_i}\frac{A_i}{A_1^*} = \frac{15}{20}(1.7732) = 1.3299$$

From which we determine the result of part (a),

$$M_t = 1.6620$$

(b) $\frac{A_s}{A_1^*} = \frac{A_s}{A_t}\frac{A_t}{A_1^*} = \frac{22}{15}1.3299 = 1.9505$; therefore from the Newton-Raphson method we find:

$$M_1 = 2.0995$$

At this Mach number we can compute the total pressure ratio across the shock

$$\frac{p_{o2}}{p_{o1}} = 0.6502 = \frac{A_1^*}{A_2^*}$$

Thus,

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{25}{15}(1.3299)(0.6502) = 1.4412$$

$M_e = 0.4571$

(c) From the various Mach numbers computed thus far, we may determine the following pressure ratios and form the string,

$$\frac{p_e}{p_a} = \frac{p_e}{p_{oe}}\frac{p_{oe}}{p_{o2}}\frac{p_{o2}}{p_{o1}}\frac{p_{o1}}{p_i} = (0.8749)(1)(0.6502)\left(\frac{1}{0.1305}\right) = 4.3591$$

Problem 11. – A jet plane uses a diverging passage as a diffuser (Figure P4.11). For a flight Mach number of 1.8, determine the range of back pressures over which a normal shock will appear in the diffuser. Ambient pressure and temperature are 25 kPa and 220 K. Find the mass flow range handled by the diffuser for the determined back pressure range. Also, the inlet and exit area are A_i = 250 cm^2, A_e = 500 cm^2. Assume isentropic flow except for the shocks. Take γ = 1.4.

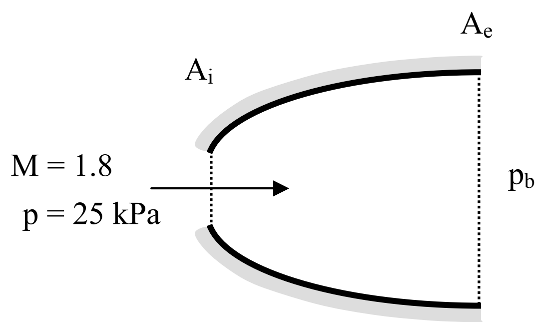


Figure P4.11

For a shock at the inlet, with M_1 = 1.8 and γ = 1.4, from the normal shock relations

$$\frac{p_{o2}}{p_{o1}} = 0.8127 = \frac{A_1^*}{A_2^*}$$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_i}\frac{A_i}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{500}{250}\right)(1.4390)(0.8127) = 2.3390$$

From this area ratio, we can determine the exit Mach number and therefore the exit static to total pressure ratio

$$M_e = 0.2574, \quad \frac{p_e}{p_{o2}} = 0.9550$$

The following pressure ratio string may be readily formed

$$p_e = \frac{p_e}{p_{o2}}\frac{p_{o2}}{p_{o1}}\frac{p_{o1}}{p_i}p_i = (0.9550)(0.8127)\frac{1}{0.1740}(25) = 111.5127 \text{ kPa}$$

Next the mass flow rate is computed

$$\dot{m} = \frac{p}{RT}AV = \frac{25}{0.287(220)}\left(250\times10^{-4}\right)\left(1.8\sqrt{1.4(287)220}\right)$$

$$= \left(0.3959\text{ kg/m}^3\right)\left(0.0250\text{m}^2\right)\left(535.1664\text{ m/s}\right)$$

$$= 5.2974\,\text{kg}/\text{s}$$

For a shock at the exit,

$$\frac{A_1}{A_1^*} = \frac{A_e}{A_i}\frac{A_i}{A_i^*} = \frac{500}{250}(1.4390) = 2.8780$$

$$M_1 = 2.5934, \quad \frac{p_1}{p_{o1}} = 0.0506, \quad \frac{p_2}{p_1} = 7.6799$$

$$p_e = \frac{p_2}{p_1}\frac{p_1}{p_{o1}}\frac{p_{o1}}{p_\infty}p_\infty = (7.6799)(0.0506)\left(\frac{1}{0.1740}\right)(25) = 55.8338\,\text{kPa}$$

The diffuser is choked so it passes the same mass flow for the back pressure range,

$$\dot{m} = 5.2974\,\text{kg/s}$$

A shock appears in the diffuser for: 55.8338 kPa < p_b < 111.5127 kPa

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Problem 12. – Air (γ = 1.4) enters a converging-diverging diffuser with a Mach number of 2.8, static pressure p_i of 100 kPa, and a static temperature of 20°C. For the flow situation shown in Figure P4.12, find the exit velocity, exit static pressure, and exit stagnation pressure.

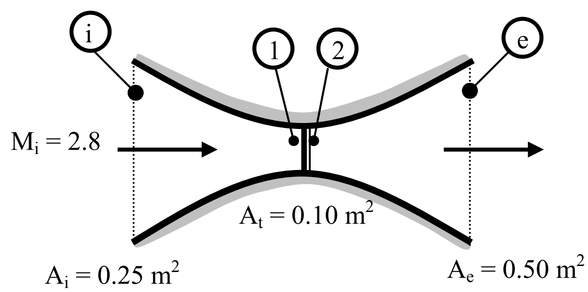

Figure P4.12

$$\text{At } M_i = 2.8, \ \frac{A_i}{A_1^*} = 3.5001, \quad \frac{p_i}{p_{o1}} = 0.0368, \quad \frac{T_i}{T_{o1}} = 0.3894$$

$$\frac{A_1}{A_1^*} = \frac{A_t}{A_i}\frac{A_i}{A_1^*} = \frac{0.10}{0.25}(3.5001) = 1.4000, \quad M_1 = 1.7632$$

From this value of M_1 we can determine the total pressure ratio across the shock

$$\frac{p_{o2}}{p_{o1}} = 0.8289 = \frac{A_1^*}{A_2^*}$$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_i}\frac{A_i}{A_1^*}\frac{A_1^*}{A_2^*} = \frac{0.5}{0.25}(3.5001)(0.8289) = 5.8025$$

From this area ratio we can compute the exit Mach number

$M_e = 0.1003$

$$\frac{p_e}{p_{o2}} = 0.9930, \quad \frac{T_e}{T_o} = 0.9980$$

$$T_o = \frac{293}{0.3894} = 752.4397\,\text{K}, \quad p_{oi} = \frac{100}{0.0368} = 2717.3913\,\text{kPa}$$

$$p_{oe} = 0.8289(2717.3913) = 2252.4457\ \text{kPa}$$

$$p_e = 0.9930(2252.4457) = 2236.6785\ \text{kPa}$$

$$V_e = M_e\sqrt{\gamma R T_e} = 0.1003\sqrt{(1.4)(287)(750.9348)} = 55.0943\,\text{m/s}$$

Problem 13. – Write a computer program that will yield values of p_2/p_1, ρ_2/ρ_1, T_2/T_1, and p_{o2}/p_{o1} for a fixed normal shock with a working fluid consisting of a perfect gas with constant $\gamma = 1.20$. Use Mach number increments of 0.05 over the range M = 1.0 to M = 2.5.

Let $b = \dfrac{\gamma - 1}{\gamma + 1}$. Then,

$$\frac{p_2}{p_1} = (b+1)M_1^2 - b$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2}{bM_1^2 + (1-b)}$$

$$\frac{T_2}{T_1} = \frac{\left[bM_1^2 + (1-b)\right]\left[(b+1)M_1^2 - b\right]}{M_1^2}$$

$$\frac{p_{o2}}{p_{o1}} = \left[\frac{M_1^2}{bM_1^2 + (1-b)}\right]^{\frac{b+1}{2b}} \left[\frac{1}{(b+1)M_1^2 - b}\right]^{\frac{1-b}{2b}}$$

M_1	p2/p1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}
1.00	1.0000	1.0000	1.0000	1.0000
1.05	1.1118	1.0923	1.0178	0.9998
1.10	1.2291	1.1873	1.0352	0.9989
1.15	1.3518	1.2848	1.0521	0.9965
1.20	1.4800	1.3846	1.0689	0.9924
1.25	1.6136	1.4865	1.0855	0.9861
1.30	1.7527	1.5902	1.1022	0.9777
1.35	1.8973	1.6957	1.1189	0.9671
1.40	2.0473	1.8027	1.1357	0.9542
1.45	2.2027	1.9110	1.1527	0.9391
1.50	2.3636	2.0204	1.1699	0.9220
1.55	2.5300	2.1308	1.1873	0.9030
1.60	2.7018	2.2420	1.2051	0.8822
1.65	2.8791	2.3539	1.2231	0.8599
1.70	3.0618	2.4663	1.2415	0.8362
1.75	3.2500	2.5789	1.2602	0.8114
1.80	3.4436	2.6918	1.2793	0.7856
1.85	3.6427	2.8048	1.2987	0.7591
1.90	3.8473	2.9177	1.3186	0.7320
1.95	4.0573	3.0304	1.3388	0.7045
2.00	4.2727	3.1429	1.3595	0.6767
2.05	4.4936	3.2549	1.3806	0.6490
2.10	4.7200	3.3664	1.4021	0.6213
2.15	4.9518	3.4773	1.4240	0.5938
2.20	5.1891	3.5876	1.4464	0.5667
2.25	5.4318	3.6971	1.4692	0.5401
2.30	5.6800	3.8058	1.4925	0.5139
2.35	5.9336	3.9135	1.5162	0.4884
2.40	6.1927	4.0203	1.5404	0.4636
2.45	6.4573	4.1261	1.5650	0.4395
2.50	6.7273	4.2308	1.5901	0.4162
M_1	p2/p1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}

Problem 14. – A converging-diverging nozzle has an area ratio (exit to throat) of 3.0. The nozzle is supplied from an air (γ = 1.4, R = 287 J/kg·K) reservoir in which the pressure and temperature are maintained at 270 kPa and 35°C, respectively. The nozzle is exhausted to a back pressure of 101 kPa. Find the nozzle exit velocity and nozzle exit-plane static pressure.

Since, $p_b/p_o = 101/270 = 0.3741 < 0.5283$, the nozzle is choked. Hence the $A_t = A^*$. So, for $A_e/A^* = 3.0$ determine the subsonic and supersonic solutions, i.e., curves 4 and 5 in Fig. 4.14. This yields $M_e = 0.1974$ and $M_e = 2.6374$.

For the subsonic solution: $p_e/p_o = 0.9732$. Thus, $p_e = (0.9732)(270) = 262.7640$ kPa, which is much larger than the given back pressure.

For the supersonic solution: $p_e/p_o = 0.04730$. Thus, $p_e = (0.0473)(270) = 12.7764$ kPa, which is far lower than the given back pressure.

The actual situation is somewhere in between these. For a shock in the exit of the nozzle (curve c in Fig 4.14), we use the shock relations at $M_e = M_1 = 2.6374$ and find $p_2/p_1 = 7.9486$. Since $p_1 = 12.7764$ kPa, $p_2 = p_e = (12.7764)(7.9486) = 101.5545$ kPa. Since this is larger than the given back pressure, this situation is also not possible.

The actual case corresponds to curve d in Fig 4.14, where oblique shock waves (refer to Fig. 4.16) occur outside the nozzle in order to compress the exiting flow to the correct pressure.

Thus, for $p_b = 101$ kPa, $p_e = 12.7764$ kPa and

$$V_e = M_e\sqrt{\gamma R T_e} = 2.6374\sqrt{1.4(287)(0.4182)308} = 599.9960\,\mathrm{m/s}$$

Problem 15. – A supersonic nozzle possessing an area ratio (exit to throat) of 3.0 is supplied from a large reservoir and is allowed to exhaust to atmospheric pressure (101 kPa). Determine the range of reservoir pressures over which a normal shock will appear in the nozzle. For what value of reservoir pressure will the nozzle be perfectly expanded, with supersonic flow at the exit plane? Find the minimum reservoir pressure to produce sonic flow at the nozzle throat. Assume isentropic flow except for shocks, with $\gamma = 1.4$.

At $A_e/A^* = 3.0$, $M_e = 0.1974$ and $p_e/p_o = 0.9732$ or $M_e = 2.6374$ and $p_e/p_o = 0.0473$.

For a shock just past the throat: $p_r = p_o = \dfrac{101}{0.9732} = 103.7813\,\text{kPa}$

For a shock at exit: $p_r = p_o = \dfrac{101}{(0.0473)(7.9486)} = 268.6393\text{kPa}$

Thus, for a shock in the nozzle: $103.7813\text{ kPa} \le p_r \le 268.6393\text{ kPa}$

For perfect isentropic expansion: $p_r = p_o = \dfrac{101}{(0.0473)} = 2135.3066$ kPa

Minimum reservoir pressure for sonic flow at nozzle throat: 103.7813 kPa

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Problem 16. – A converging-diverging nozzle with an area ratio (exit to throat) of 3.0 exhausts air ($\gamma = 1.4$) from a large high-pressure reservoir to a region of back pressure p_b. Under a certain operating condition, a normal shock is observed in the nozzle at an area equal to 2.2 times the throat area. What percent of decrease in back pressure would be necessary to rid the nozzle of the normal shock?

For $A_s/A^* = 2.2$, $M_s = M_1 = 2.3034$. At this Mach number from the shock tables we find:

$$\frac{p_{o2}}{p_{o1}} = \frac{A_1^*}{A_2^*} = 0.5818$$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_1^*}\frac{A_1^*}{A_2^*} = (3.0)(1)(0.5818) = 1.7454$$

From this area ratio we find, $M_e = 0.3577$ from which $p_e/p_{o2} = 0.9154$. Thus,

$$\frac{p_e}{p_r} = \frac{p_e}{p_{o1}} = \frac{p_e}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.9154)(0.5818) = 0.5326$$

Now for a shock at the exit, i.e., $A_s/A^* = A_e/A_t = 3.0$: $M_1 = = 2.6374$ and in turn Me = 0.5005.

$$\frac{p_e}{p_r} = \frac{p_e}{p_{o1}} = \frac{p_1}{p_{o1}}\frac{p_e}{p_2}\frac{p_2}{p_1} = (0.0473)(1)(7.9485) = 0.3760$$

$$\%\text{ reduction} = \frac{0.5326 - 0.3760}{0.5326}100 = 29.4097\%$$

Problem 17. – Due to variations in fuel flow rate, it is found that the stagnation pressure at the inlet to a jet-engine nozzle varies with time according to:

$$p_o = 200[1 + 0.1 \sin (\pi/4)t],$$

with t in seconds and p_o in kilopascals. Determine the resultant variation in nozzle flow rate, nozzle exhaust velocity, and exit-plane static pressure. The nozzle area ratio (exit to throat) is 2.0 to 1, and the inlet stagnation temperature is 600 K. Assume negligible inlet velocity. The nozzle exhausts to an ambient pressure of 30 kPa; $\gamma = 1.4$; nozzle exit area is 0.3 m^2; R = 0.3 kJ/kg · K.

$$\dot{m}_{th} = \rho_t A_t V_t = \frac{p_t}{RT_t} A_t M_t \sqrt{\gamma R T_t} = \frac{\left(\frac{p}{p_o}\right)_t p_o}{R\left(\frac{T}{T_o}\right)_t T_o} \frac{A_e}{\left(\frac{A_e}{A_t}\right)} M_t \sqrt{\gamma R \left(\frac{T}{T_o}\right)_t T_o}$$

$$= \frac{(0.5283)p_o}{(0.3)(0.8333)(600)} \left(\frac{0.3}{2.0}\right) 1.0 \sqrt{(1.4)(300)(0.8333)(600)}$$

$$= 0.2421 p_o = 48.4200 + 4.8420 \sin\left(\frac{\pi}{4} t\right)$$

Hence, the stagnation pressure varies from 48.4200 – 4.8420 = 43.5780 kg/s to 48.4200 + 4.8420 = 53.2620 kg/s.

The stagnation pressure varies from 200 – 20 = 180 kPa to 200 + 20 = 220 kPa.

For a shock at the exit, for $A_e/A^* = 2.0$, we find $M_e = M_1 = 2.1972$. From which we obtain and $(p_1/p_{o1}) = 0.0939$ and $(p_2/p_1) = 5.4656$. Thus,

$$p_r = p_o = \frac{30}{(0.0939)(5.4656)} = 58.4545 \text{kPa}$$

Hence, the exit velocity is constant,

$$V_e = M_e \sqrt{\gamma R \left(\frac{T}{T_o}\right)_e T_o} = 2.1972 \sqrt{(1.4)(300)(0.5088)(600)} = 786.7621 \text{m/s}$$

$$p_e = (0.0939) p_o = 18.7800 + 1.8780 \sin\left(\frac{\pi}{4} t\right)$$

Problem 18. – Helium enters a converging-diverging nozzle with a negligible velocity; stagnation pressure is 500 kPa and stagnation temperature is 300 K. The nozzle throat area is 50 cm^2, and the exit area is 300 cm^2. Determine the range of nozzle back pressures over which a normal shock will appear in the nozzle. Also, find the nozzle exit velocity if the nozzle exhausts into a vacuum.

For $\gamma = 5/3$ and at an area ratio $(A/A^*) = 300/50 = 6.0$, we find,

$M = 0.0943$ and $p/p_o = 0.9926$

$M = 4.1051$, $p/p_o = 0.008878$, $p_2/p_1 = 20.8145$ and $T/T_o = 0.1511$.

A normal shock will be in the nozzle for $(0.008878)(20.8145)(500) = 92.3956$ kPa $< p_b <$ $(0.9926)(500) = 496.3000$ kPa.

$$V_e = M_e\sqrt{\gamma R\left(\frac{T}{T_o}\right)_e T_o} = 4.1051\sqrt{\left(\frac{5}{3}\right)(2077)(0.1511)(300)} = 1626.1448\,\text{m/s}$$

Problem 19. – A jet plane uses a diverging passage as a diffuser (Figure P4.19). For a flight Mach number of 1.92, determine the range of back pressures over which a normal shock will appear in the diffuser. Ambient pressure and temperature are 70 kPa and 270 K. Find the mass flow rates handled by the diffuser for the determined back pressure ranges, with $A_{inlet} = 100$ cm^2 and $A_{exit} = 200$ cm^2. Assume isentropic air flow ($\gamma = 1.4$, $R = 287$ J/kg·K) except for across the shocks.

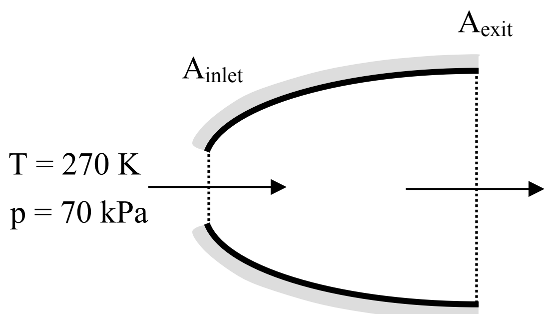


Figure P4.19

When the shock is at the inlet: i (the inlet) = 1 (the upstream location of the shock)

At $M_1 = 1.92$, $p_1/p_{o1} = 0.1447$, $A_1^*/A_2^* = 0.7581$ and $A_1/A_1^* = 1.5804$. Thus,

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{200}{100}\right)(1.5804)(0.7581) = 2.3962$$

From this area ratio we find, $M_e = 0.2507$ and $p_e/p_{o2} = 0.9572$. Now $p_1 = 70$ kPa, thus

$$p_e = p_b = \frac{p_e}{p_{o2}}\frac{p_{o1}}{p_1}\frac{A_1^*}{A_2^*}p_1 = \frac{(0.9572)(0.7581)}{(0.1447)}70 = 351.0417\,\text{kPa}$$

When the shock is at the exit: e (the exit) = 2 (the downstream location of the shock)

At $M_i = 1.92$, $A_i/A_1^* = 1.5804$. Thus

$$\frac{A_1}{A_1^*} = \frac{A_1}{A_i}\frac{A_i}{A_1^*} = \left(\frac{200}{100}\right)(1.5804) = 3.1608$$

From this area ratio we find, $M_1 = 2.6926$, $p_2/p_1 = 8.2918$ and $p_1/p_{o1} = 0.04344$. Because the flow is isentropic from i to 1 we may write,

$$p_1 = \frac{p_1}{p_{o1}}\frac{p_{oi}}{p_i}p_i = \frac{0.04344}{0.1447}70 = 21.0145\,\text{kPa}$$

$$p_b = p_e = p_2 = \frac{p_2}{p_1}p_1 = (8.2918)(21.0145) = 174.2481\,\text{kPa}$$

A normal shock will be in the diffuser for 174.2481 kPa $\leq p_b \leq$ 351.0417 kPa

$$\dot{m}_i = \rho_i A_i V_i = \frac{p_i}{RT_i}A_i M_i\sqrt{\gamma RT_i} = \frac{70}{(0.287)(270)}\left(100\text{x}10^{-4}\right)(1.92)\sqrt{(1.4)(287)(270)}$$

$$= 5.7127\,\text{kg/s}$$

Problem 20. – For the converging-diverging nozzle shown in Figure P4.20, find the range of back pressures for which $p_e > p_b$, the range of back pressures for which $p_e < p_b$, and the range of back pressures over which the nozzle is choked. Take $\gamma = 1.4$.

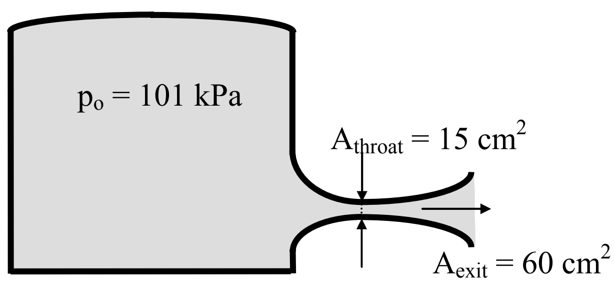


Figure P4.20

For an area ratio $A_e/A_t = A_e/A^* = 60/15 = 4.0$,

Supersonic case with shock at exit:

$M_e = 2.9402$, $p_e/p_o = 0.0298$ and $p_2/p_1 = 9.9188$

$$p_b = \frac{p_2}{p_1}\frac{p_1}{p_{o1}}p_{o1} = (9.9188)(0.0298)(101) = 29.8536\,\text{kPa}$$

Subsonic case with shock just downstream of throat:

$M_e = 0.1465$ and $p_e/p_o = 0.9851$

$p_b = (0.9851)(101) = 99.4951$ kPa

For perfectly expanded flow in nozzle:

$p_b = (0.0298)(101) = 3.0098$ kPa

So, $p_e > p_b$ for all $p_b < 3.0098$ kPa, whereas, $p_e < p_b$ for 3.0098 kPa $< p_b < 29.8536$ kPa. The nozzle is choked for all $p_b \le 99.4951$ kPa.

Problem 21. – Nitrogen ($\gamma = 1.4$, $R = 296.8$ J/kg·K) expands in a converging-diverging nozzle from negligible velocity, a stagnation pressure of 1 MPa, and a stagnation temperature of 1000 K to supersonic velocity in the diverging portion of the nozzle. If the area ratio of the nozzle is 4.0, determine the back-pressure necessary for a normal shock to position itself at an area equal to twice the throat area. For this condition, find the nozzle exit velocity.

For $A_s/A_1^* = 2.0$, $M_s = M_1 = 2.1972$. At this Mach number from the shock relations we find:

$$\frac{p_{o2}}{p_{o1}} = \frac{A_1^*}{A_2^*} = 0.6294$$

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_t}{A_1^*}\frac{A_1^*}{A_2^*} = (4.0)(1)(0.6294) = 2.5176$$

From this area ratio we find, $M_e = 0.2377$ from which $p_e/p_{o2} = 0.9614$ and $T_e/T_o = 0.9888$. Thus,

$$\frac{p_e}{p_r} = \frac{p_e}{p_{o1}} = \frac{p_e}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.9614)(0.6295) = 0.6052$$

$p_b = 0.6052(1\text{MPa}) = 605.2$ kPa

$T_e = 0.9888(1000) = 988.8\text{K}$

$$V_e = M_e\sqrt{\gamma R T_e} = 0.2377\sqrt{(1.4)(296.8)(988.8)} = 152.3630\,\text{m/s}$$

Problem 22. – (a) Develop a relation for the upstream Mach number, M_1, in terms of the downstream Mach number, M_2. (b) Use the result from (a) and Eq. (4.12) to prove that $M_2^2 = [(\gamma+1)/2\gamma][(\gamma-1)/(\gamma+1) + p_1/p_2]$.

Let b = (γ+1)/(γ-1), therefore b+1 = 2γ/(γ-1) and b-1 = 2/(γ-1).

(a) Equation (4.9) may be written as

$$M_2^2 = \frac{M_1^2 + (b-1)}{(b+1)M_1^2 - 1}$$

Expand this and rearrange to get

$$(b+1)M_1^2 M_2^2 = M_1^2 + M_2^2 + b - 1$$

Note by interchanging the subscripts the relation is unchanged, therefore it is obvious that

$$M_1^2 = \frac{M_2^2 + (b-1)}{(b+1)M_2^2 - 1} = \frac{M_2^2 + \dfrac{2}{\gamma-1}}{\left(\dfrac{2\gamma}{\gamma-1}\right)M_2^2 - 1}$$

The result is also apparent from Fig. 4.10 in which we may observe that the curve is symmetrical about the line $M_2 = M_1$.

(b) Equation (4.12) can be written as

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1} = \frac{b+1}{b}M_1^2 - \frac{1}{b} = \left(\frac{b+1}{b}\right)\left[\frac{M_2^2 + (b-1)}{(b+1)M_2^2 - 1}\right] - \frac{1}{b}$$

$$= \frac{b}{(b+1)M_2^2 - 1}$$

or

$$M_2^2 = \left(\frac{b}{b+1}\right)\left(\frac{p_1}{p_2} + \frac{1}{b}\right) = \left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{\gamma-1}{\gamma+1} + \frac{p_1}{p_2}\right)$$

Problem 23. – Prove that the Rankine-Hugoniot relation reduces to the equation for an isentropic process for very weak shocks. Hint: start from Eq. (4.16b) and replace p_2 with p + dp and p_1 with p. Repeat this for the densities. Then use the expansion technique that was employed in Example 4.1. Note to properly use the expansion approach we must first express the term to be expanded as 1 + (small quantity).

Let b = (γ+1)/(γ-1), therefore b+1 = 2γ/(γ-1) and b-1 = 2/(γ-1). Thus, Eq. (4.16b) may be written as

$$\frac{\rho_2}{\rho_1} = \frac{b\dfrac{p_2}{p_1}+1}{b+\dfrac{p_2}{p_1}}$$

Now replace the downstream terms with the upstream value + a differential and rearrange the result to get

$$1+\frac{d\rho}{\rho} = \frac{1+\left(\dfrac{b}{b+1}\right)\dfrac{dp}{p}}{1+\left(\dfrac{1}{b+1}\right)\dfrac{dp}{p}} = \left[1+\left(\frac{b}{b+1}\right)\frac{dp}{p}\right]\left[1+\left(\frac{1}{b+1}\right)\frac{dp}{p}\right]^{-1}$$

$$= \left[1+\left(\frac{b}{b+1}\right)\frac{dp}{p}\right]\left[1-\left(\frac{1}{b+1}\right)\frac{dp}{p}\right] \approx 1+\left(\frac{b}{b+1}-\frac{1}{b+1}\right)\frac{dp}{p}$$

$$= 1+\left(\frac{\gamma+1}{2\gamma}-\frac{\gamma-1}{2\gamma}\right)\frac{dp}{p} = 1+\frac{1}{\gamma}\frac{dp}{p}$$

Thus,

$$\frac{dp}{p} = \gamma\frac{d\rho}{\rho}$$

Integration gives the isentropic relation $p = C\rho^{\gamma}$

Problem 24. – The back pressure to reservoir pressure ratio is 0.7 for a C-D nozzle, with an exit to throat area ratio of 2.0. Use the procedure when the shock location is not specified, i.e., the direct approach to determine the location of a normal shock for a ratio of specific heats equal to 1.3. Repeat the problem for γ = 5/3. Draw a conclusion regarding shock location and the value of γ.

The following table showing the calculation results was prepared from a simple spreadsheet program

step	γ	1.3	1.4	1.67
1	M_e	0.4128	0.4067	0.3919
2	p_e/p_{o2}	0.8964	0.8923	0.8827
3	p_{o2}/p_{o1}	0.7809	0.7845	0.793
4	M_1	1.8397	1.8627	1.9221
5	A_s/A_t	1.5349	1.5101	1.4573

As may be seen as γ is increased the shock moves upstream.

Problem 25. – The back-pressure to reservoir pressure ratio is 0.7 for a C-D nozzle, with an exit to throat area ratio of 2.0. Use the procedure for the situation when the shock location is specified, i.e., the trial and error approach to determine the location of a normal shock for a ratio of specific heats equal to 1.4. To start the calculations assume the shock is at the exit of the nozzle.

The following table summarizes the calculations for each trial.

step	trial	1	2	3
0	A_s/A_t	2.0	1.467	1.508
1	M_1	2.197	1.825	1.861
2	p_{o2}/p_{o1}	0.6295	0.8015	0.7852
3	A_1^*/A_2^*	0.6295	0.8015	0.7852
4	A_e/A_2^*	1.2590	1.6030	1.5704
5	M_e	0.5473	0.3960	0.4062
6	p_e/p_{o2}	0.8158	0.8975	0.8926
7	p_e/p_{o1}	0.53135	0.7193	0.7009
8	% error	-26.6	2.8	0.1
9	A_s/A_t	1.467	1.508	1.51

Problem 26. – A converging-diverging supersonic diffuser is to be used at Mach 3.0. The diffuser is to use a variable throat area so as to swallow the starting shock. What percent of increase in throat area will be necessary? Solve for air (γ = 1.4) and for helium (γ = 5/3) as working fluids.

Air:

With no shock from Eq.(3.23) at M = 3.0, $\dfrac{A_i}{A_{throat}} = 4.2346$

With shock at inlet, M_1 = 3.0 and from Eq.(4.9), M_2 = 0.4752. Using this Mach number downstream of the shock in Eq.(3.23), we find $A_2 / A_2^* = 1.3904$.

Throat area must be increased slightly more than:

$$\Delta A = \frac{\dfrac{A_i}{1.3904} - \dfrac{A_i}{4.2346}}{\dfrac{A_i}{4.2346}} \times 100 = \left(\frac{4.2346}{1.3904} - 1 \right) 100 = 204\%$$

Helium:

With no shock from Eq.(3.23) at M = 3.0, $\dfrac{A_i}{A_{throat}} = 3.0000$

With shock at inlet, M_1 = 3.0 and from Eq.(4.9), M_2 = 0.5222. Using this Mach number downstream of the shock in Eq.(3.23), we find $A_2 / A_2^* = 1.2819$.

Throat area must be increased slightly more than:

$$\Delta A = \frac{\dfrac{A_i}{1.2819} - \dfrac{A_i}{3.0000}}{\dfrac{A_i}{3.0000}} \times 100 = \left(\frac{3.0000}{1.2819} - 1 \right) 100 = 134\%$$

Problem 27. –A supersonic wind tunnel is to be constructed as shown in Figure 4.27, with air (γ = 1.4, R = 287 J/kg·K) at atmospheric pressure passing through a converging-diverging nozzle into a constant-area test section and then into a large vacuum tank. The test run is started with a pressure 0 kPa in the tank. How long can uniform flow conditions be maintained in the test section (i.e., how long will it be before the tank pressure rises to a value such that a shock will appear in the test section)? Assume the test section to be circular, 10 cm in diameter, with a design Mach number of 2.4. The tank volume is 3 m^3, with atmospheric conditions of 101 kPa and 20°C. Assume the air to be brought to rest adiabatically in the tank.

For a shock at the nozzle exit

$$p_b = p_2 = \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{o1} = (6.5533)(.06840)(101) = 45.272 \text{ kPa}$$

Tunnel will run until

$$m_{tank} = \frac{p\forall}{RT} = \frac{\left(45.2728 \text{ kN/m}^2\right)\left(3 \text{ m}^3\right)}{(0.287 \text{ kJ/kg}\cdot\text{k})(293 \text{ K})} = 1.6151 \text{ kg}$$

The mass flow rate is constant while tunnel is running, so

$$\dot{m} = \rho AV = \rho AMa = \left(\frac{p}{RT}\right)AM\sqrt{\gamma RT} = \left[\frac{p_o\left(\frac{p}{p_o}\right)}{RT_o\left(\frac{T}{T_o}\right)}\right]AM\sqrt{\gamma RT}$$

$$= \left[\frac{101(0.06840)}{0.287(0.4647)293}\right]\left(\frac{\pi}{4}100\times10^{-4}\right)2.4\sqrt{1.4(287)(0.4647)(293)}$$

$$= (0.1768)(0.007854)(561.3533) = 0.7794 \text{ kg/s}$$

$$\text{Time to run } = \frac{m}{\dot{m}} = \frac{1.6151}{0.7794} = 2.0722 \text{ s}$$

Problem 28. – Repeat Problem 27 but assume that there is a diffuser of area ratio 2 to 1 between the test section and the tank.

Now at M = 2.4 we can find from Eq.(3.23) that $A/A_1^* = 2.4031$. Furthermore, from Eqs.(4.15) and (4.21): $p_{o2}/p_{o1} = A_1^*/A_2^* = 0.5401$. Therefore, since $A_e/A_2 = 2.0$, then

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = (2.0)(2.403)(0.5401) = 2.5957$$

Using the area ratio-Mach number numerical procedure, the subsonic solution gives for this area ratio

$$M_e = 0.2301$$

Hence,

$$\frac{p_e}{p_{o2}} = 0.9638, \quad p_e = \left(\frac{p_e}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right)p_{o1} = (0.9638)(0.5401)101 = 52.5754 \text{ kPa}$$

$$\text{Time} = \frac{52.5754}{45.2728}2.0722 = 2.4064 \text{ s}$$

Chapter Five

MOVING NORMAL SHOCK WAVES

Problem 1. – A projectile moves down a gun barrel with a velocity of 500 m/s (Figure P5.1). (a) Calculate the velocity of the normal shock that would precede the projectile. Assume the pressure in the undisturbed air (γ = 1.4, R = 287 J/kg·K) to be 101 kPa and the temperature to be 25°C. (b) How fast would the projectile have to be moving in order for the shock velocity to be two times the projectile velocity?

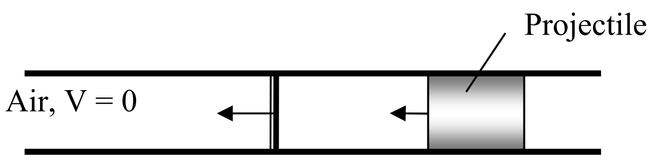


Figure P5.1

(a) From Eq. (5.10),

$$S = \frac{(\gamma+1)V}{4} + \sqrt{\frac{(\gamma+1)^2 V^2}{16} + a_1^2}$$

$$a_1 = \sqrt{1.4(287)(298)} = 346.0295 \text{ m/s}$$

$$S = (0.6)500 + \sqrt{300^2 + 346.0295^2} = 757.9699 \text{ m/s}$$

(b) For this part of the problem S = 2V. This is inserted into Eq. (5.9)

$$V = \frac{2(2V)}{(\gamma+1)}\left[1 - \left(\frac{a_1}{2V}\right)^2\right]$$

Cancellation and rearrangement brings

$$V = \frac{a_1}{\sqrt{3-\gamma}} = \frac{346.0295}{1.2649} = 273.5603 \text{ m/s}$$

Problem 2. – A normal shock moves into still air (γ = 1.4, R = 287 J/kg·K) with a velocity of 1,000 m/s. The motionless air is at 101 kPa and 20°C; calculate the following:

(a) the velocity of the air flow behind the wave,
(b) the static pressure behind the wave, and

(c) the stagnation temperature behind the wave

(a) $$M_1 = \frac{S}{a_1} = \frac{S}{\sqrt{\gamma RT}} \frac{1000}{\sqrt{1.4(287)293}} = 2.9146$$

Use this Mach number in the shock relations, to determine that

$$\frac{\rho_2}{\rho_1} = 3.7769, \quad \frac{p_2}{p_1} = 9.7440, \quad \frac{T_2}{T_1} = 2.5799$$

Hence,

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{S}{S-V} = \frac{1000}{1000-V} = 3.7769$$

From which we find, V = 735.2326 m/s.

(b) $$\frac{p_2}{p_1} = 9.7440, \quad p_2 = 9.7440(101) = 984.1440 \text{ kPa}$$

(c) $$\frac{T_2}{T_1} = 2.5799, \quad T_2 = 2.5799(293) = 755.9107 \text{ K}$$

$$a_2 = \sqrt{\gamma RT_2} = \sqrt{1.4(287)755.9} = 551.1124 \text{ m/s}$$

$$T_{o2} = T_1\left[1 + \frac{\gamma-1}{2}\left(\frac{V}{a_2}\right)^2\right] = 755.9107\left[1 + (0.2)\left(\frac{735.2326}{551.1124}\right)^2\right] = 1024.9834\text{K}$$

Problem 3. – A normal shock is observed to move through a constant-area tube into air (γ = 1.4, R = 287 J/kg·K) at rest at 25°C (Figure P5.3). The velocity of the air behind the wave is measured to be 150 m/s. Calculate the shock velocity.

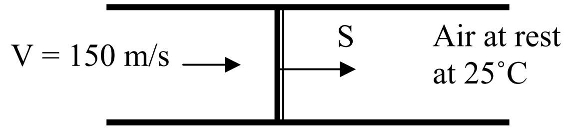


Figure P5.3

From Eq. (5.10),

$$S = \frac{(\gamma+1)V}{4} + \sqrt{\frac{(\gamma+1)^2 V^2}{16} + a_1^2}$$

$$a_1 = \sqrt{\gamma RT_2} = \sqrt{1.4(287)298} = 346.0295\,\text{m/s}$$

$$S = 90.0 + 357.5 = 447.5\,\text{m/s}$$

Problem 4. – A piston in a tube is suddenly accelerated to a velocity of 25 m/s causing a normal shock to move into helium (γ = 5/3, R = 2077 J/kg·K) at rest in the tube and at a temperature of 27 C in the tube. One second later, the piston is suddenly accelerated from 25 to 50 m/s causing a second shock to move down the tube. How much time will elapse from the initial acceleration of the piston to the intersection of the two shocks?

First shock:

$$a_1 = \sqrt{\gamma RT_1} = \sqrt{1.667(2077)300} = 1{,}019.0682\text{m/s}$$

From Eq. (5.10)

$$S_F = \frac{(\gamma+1)V_2}{4} + \sqrt{\frac{(\gamma+1)^2 V_2^2}{16} + a_1^2} = \frac{8}{12}(25) + \sqrt{\left[\frac{8}{12}(25)\right]^2 + 1019.0682^2}$$
$$= 1035.8711\,\text{m/s}$$

$$M_1 = \frac{S_F}{a_1} = \frac{1035.8711}{1019.0682} = 1.0165$$

At this Mach number from Eq.(4.11) or (5.13)

$$\frac{T_2}{T_1} = 1.0164$$; thus, T_2 = (1.0164)300 = 304.9200K

So

$$a_2 = \sqrt{\gamma RT_1} = \sqrt{1.667(2077)304.92} = 1{,}027.3906\text{m/s}$$

Second shock:

Following Example 5.4, we may write

$$S_S = \frac{(\gamma+1)V_3 + (3-\gamma)V_2}{4} + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)(V_3 - V_2)\right]^2 + a_2^2}$$

Hence, with $V_2 = 25$ m/s and $V_3 = 50$ m/s

$$S_S = \frac{(8)50 + (4)25}{12} + \sqrt{\left[\left(\frac{8}{12}\right)(50-25)\right]^2 + 1027.3906^2} = 1069.1924\,\text{m/s}$$

$$1035.8711\Delta t = 1069.1924(\Delta t - 1)$$

$$\Delta t = 32.0874\text{s}$$

Problem 5. – Air ($\gamma = 1.4$, R = 287 J/kg·K) at 100 kPa and 290 K is flowing in a constant-area tube with a velocity of 100 m/s (Figure P5.5). Suddenly the end of the tube is closed, which causes a normal shock to propagate back through the airstream. Find the absolute velocity of this shock.

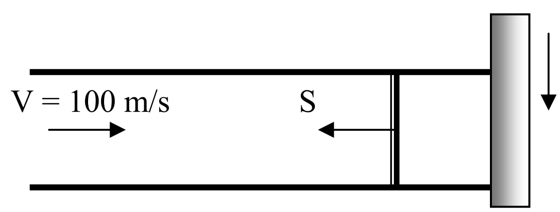
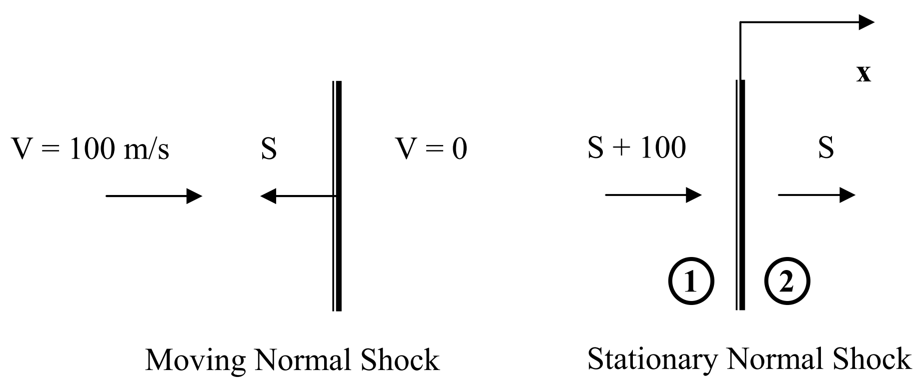


Figure P5.5

First fix the shock



To compute the shock speed use Eq.(5.10). However, the speed S that appears in the expression must be replaced with S + V to agree with the current problem. Accordingly we may write:

$$S + V = \left(\frac{\gamma+1}{4}\right)V + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V\right]^2 + a_1^2}$$

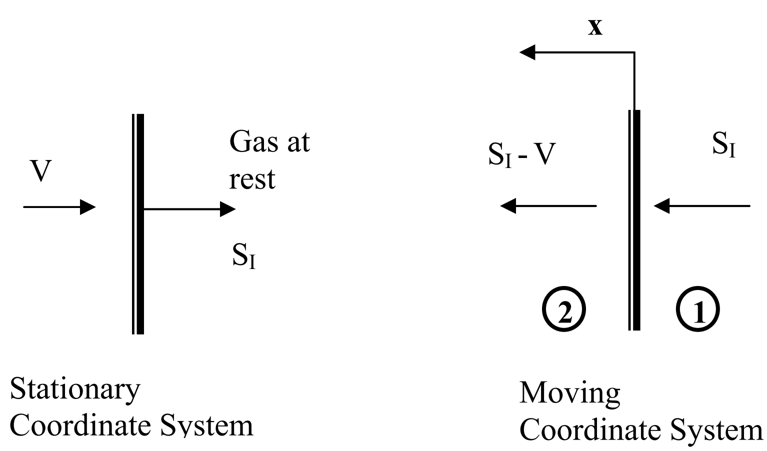
The speed of sound of the gas in front of the moving shock wave, i.e., a_1 is required. Accordingly we may write

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)(287)(290)} = 341.3532\,\text{m/s}$$

$$S = \left(\frac{\gamma-3}{4}\right)V + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V\right]^2 + a_1^2} = (-0.40)100 + \sqrt{60^2 + 341.3532^2} = 306.5862\,\text{m/s}$$

Problem 6. – A normal shock traveling at 1,000 m/s into still air (γ = 1.4, R = 287 J/kg·K) at 0°C and 101 kPa reflects from a plane wall. Determine the velocity of the reflected shock. Compare the pressure ratio across the reflected shock with that across the incident shock. Find the stagnation pressure that would be measured by a stationary observer behind the reflected wave.

Incident shock: First we must immobilize the shock by redefining a coordinate system that moves with the shock



$$a_1 = \sqrt{\gamma R T_1} = 20.0449\sqrt{273} = 331.1969\,\text{m/s}$$

$$M_1 = \frac{S_I}{a_1} = \frac{1000}{331.1969} = 3.0194$$

From the normal shock relations:

$$M_2 = 0.47405,\ \frac{\rho_2}{\rho_1} = 3.8749\ \ \frac{T_2}{T_1} = 2.7019\ \ \frac{p_2}{p_1} = 10.4696 .$$

Thus,

$$T_2 = (2.7019)273 = 737.6187\text{K} ,$$

$$a_2 = \sqrt{(1.4)(287)(737.6187)} = 544.4035\ \text{m/s} .$$

$$M_2 = \frac{S_I - V}{a_2}$$

$$V = S_I - M_2 a_2 = 1000 - (0.47405)544.4035 = 741.9255\ \text{m/s}$$

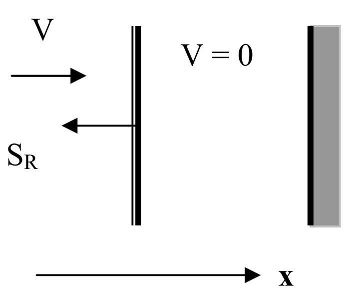
Alternately,

$$\frac{\rho_2}{\rho_1} = \frac{S_I}{S_I - V} = 3.8749 = \frac{1000}{1000 - V}$$

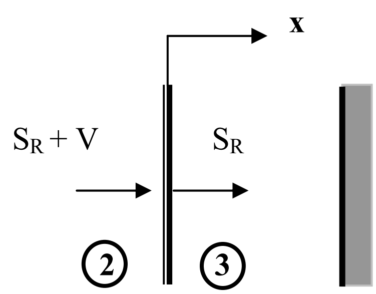
Solve to obtain

V = 741.9288 m/s

Reflected shock: Again the first step is to fix the moving shock by redefining the coordinate system.



Moving Reflected Shock



Stationary Coordinate System
for the Reflected Shock

For this configuration, the reflected shock speed is computed from

$$S_R + V = \left(\frac{\gamma+1}{4}\right)V + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V\right]^2 + a_2^2}$$

where V = 741.9288 m/s and a_2 = 544.4035 m/s; therefore,

$$S_R = \left(\frac{\gamma-3}{4}\right)V + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V\right]^2 + a_2^2}$$

$$= -(0.4)741.9288 + \sqrt{[(0.6)741.9288]^2 + 544.4035^2}$$

$$= 406.4640\,\mathrm{m/s}$$

From Eq.(5.24)

$$\frac{p_3}{p_2} = \frac{\left(\frac{3\gamma-1}{\gamma+1}\right)\left(\frac{p_2}{p_1}\right) - \left(\frac{\gamma-1}{\gamma+1}\right)}{1 + \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{p_2}{p_1}\right)} = \frac{\left(\frac{4}{3}\right)10.4696 - \left(\frac{1}{6}\right)}{1 + \left(\frac{1}{6}\right)10.4696} = 5.0248$$

Behind reflected wave, because the velocity = 0

$$p_3 = p_o = \left(\frac{p_3}{p_2}\right)\left(\frac{p_2}{p_1}\right)p_1 = 10.4696(5.0248)101 = 5.3133\text{ MPa}$$

Problem *7*. – Under a certain operating condition, the piston speed in an auto engine is 10 m/s. Approximate engine knock as the occurrence of a normal shock wave traveling at 1000 m/s downward, as shown in Figure P5.7, into the unburned mixture at 700 kPa and 500 K. Determine the pressure acting on the piston face after the shock reflects from it. Assume the gas has the properties of air (R = 287 J/kg·K) and acts as a perfect gas, with $\gamma = 1.4$.

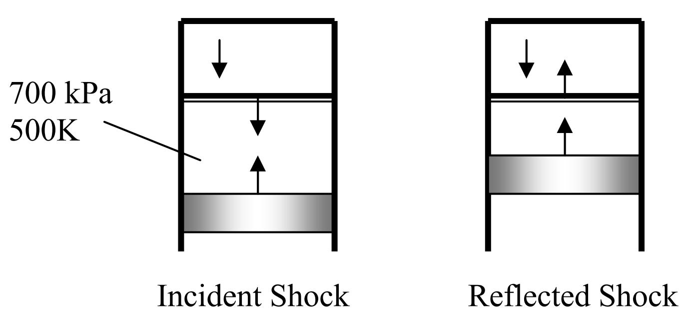
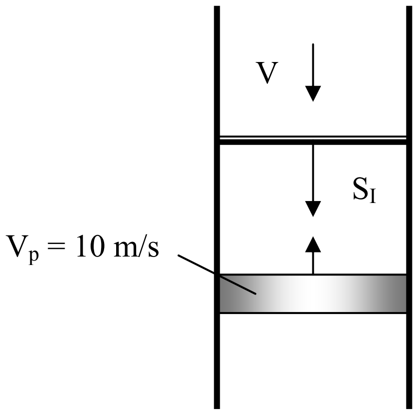
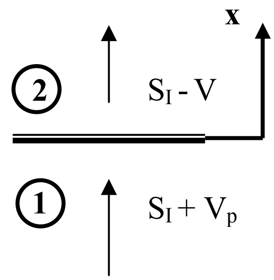


Figure P5.7

Incident shock: We must first consider the moving incident shock and redefine the coordinate system in order to produce a steady flow problem.



Incident Shock in Stationary Coordinate System



Incident Shock in Moving Coordinate System

Now $V_1 = S_I + V_p = 1000 + 10 = 1010\,m/s$ and

$a_1 = \sqrt{(1.4)(287)(500)} = 448.2187\,m/s.$

Thus,

$$M_1 = \frac{1010}{448.2187} = 2.2534\,.$$

Therefore, from the normal shock relations,

$$M_2 = 0.5401,\ \frac{\rho_2}{\rho_1} = 3.0232\ \ \frac{T_2}{T_1} = 1.9044\ \ \frac{p_2}{p_1} = 5.7574$$

$$T_2 = (1.9044)500 = 952.2000K$$

$$a_2 = \sqrt{(1.4)(287)(952.2)} = 618.5418\,m/s\,.$$

From the continuity equation across the stationary shock

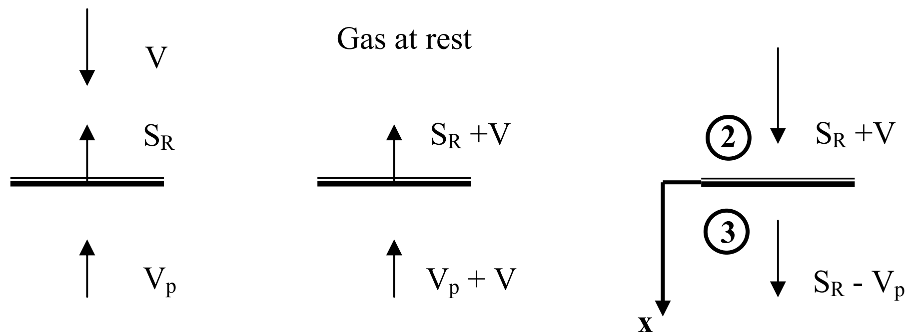
$$\frac{\rho_2}{\rho_1} = \frac{S_I + V_p}{S_I - V} = 3.0232 = \frac{1010}{1000 - V}$$

Therefore,V = 665.9169 m/s. Alternately,

$$M_2 = \frac{S_I - V}{a_2}$$

$$V = S_I - M_2 a_2 = 1000 - (0.5401)618.5418 = 665.9256\,\mathrm{m/s}$$

Reflected shock: Fix reflected shock:



Reflected Shock in a Stationary Coordinate System

Intermediate Step

Reflected Shock in a Moving Coordinate System

From the intermediate step, which results in a normal shock moving into a fluid at rest (the fundamental problem), we may use the equations of Section 5.2. However, we must replace the shock speed, S, in those relations, with $S_R + V$ and we must replace the gas speed behind the shock V with $V_p + V$. Accordingly, we may rewrite Eq.(5.10) as

$$S_R + V = \left(\frac{\gamma+1}{4}\right)(V_p + V) + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)(V_p + V)\right]^2 + a_2^2}$$

Now V = 665.9256 m/s, V_p = 10 m/s and a_2 = 618.5418 m/s, therefore

$$S_R = -665.9256 + (0.6)(675.9256) + \sqrt{[(0.6)(675.9256)]^2 + 618.5418^2}$$

$$= 479.2710\,\mathrm{m/s}$$

From Eq.(5.24)

$$\frac{p_3}{p_2} = \frac{\left(\frac{3\gamma-1}{\gamma+1}\right)\left(\frac{p_2}{p_1}\right) - \left(\frac{\gamma-1}{\gamma+1}\right)}{1 + \left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{p_2}{p_1}\right)} = \frac{\left(\frac{4}{3}\right)5.7574 - \left(\frac{1}{6}\right)}{1 + \left(\frac{1}{6}\right)5.7574} = 3.8324$$

Pressure on piston face $= p_1\left(\frac{p_2}{p_1}\right)\left(\frac{p_3}{p_2}\right) = (700\text{ kPa})(5.7574)(3.8324) = 15.445\,\text{MPa}$

or

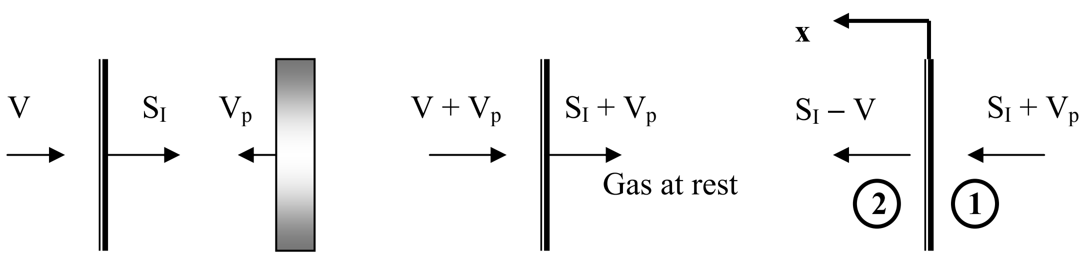
$= 152.9234$ atm

Problem 8. – A normal shock moves down a tube with a velocity of 600 m/s into a gas with static p = 50 kPa and static temperature of 300 K. At the end of the tube, a piston is moving with a velocity of 60 m/s, as shown in Figure P5.8. Calculate the velocity of the reflected wave and the static pressure behind the reflected wave. Assume the gas has the properties of air ($\gamma = 1.4$, R = 287 J/kg·K).

Incident Normal Shock　　　　Reflected Normal Shock

Figure P5.8

Incident Shock: As usual we perform the coordinate transformation to fix the incident shock. Because the gas in front of the shock is moving it is helpful to perform an intermediate step in which this gas is brought to rest. In this way the equations pertaining to a normal shock moving into a stationary gas may be transformed to this problem.



Incident Normal Shock in a Stationary Coordinate System　　Intermediate Step　　Incident Normal Shock in a Moving Coordinate System

$$a_1 = \sqrt{(1.4)(287)(300)} = 347.1887\text{ m/s},\ \ M_1 = \frac{S_I + V_p}{a_1} = \frac{600+60}{347.1887} = 1.9010$$

$$\frac{p_2}{p_1} = 4.0494,\quad \frac{T_2}{T_1} = 1.6087,$$

$$T_2 = (1.6087)300 = 482.6100\text{ K},\ \ a_2 = \sqrt{(1.4)(287)(482.61)} = 440.3552\text{ m/s}$$

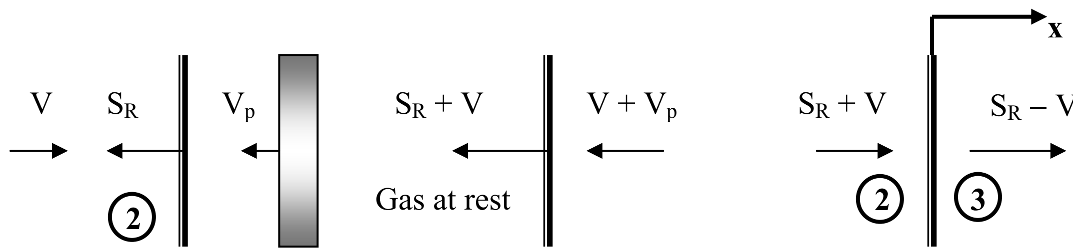
Now from Eq.(5.9) with S replaced by $S_I + V_p$ and V replaced by $V + V_p$ (see the intermediate step), i.e.,

$$\frac{V}{S} = \frac{2}{\gamma+1}\left[1-\left(\frac{a_1}{S}\right)^2\right] \text{ becomes}$$

$$\frac{V+V_p}{S_I+V_p} = \frac{2}{\gamma+1}\left[1-\left(\frac{a_1}{S_I+V_p}\right)^2\right], \text{ or}$$

$$V = -V_p + \frac{2(S_I+V_p)}{\gamma+1}\left[1-\left(\frac{a_1}{S_I+V_p}\right)^2\right] = -60 + \frac{2(660)}{2.4}\left[1-\left(\frac{347.1887}{660}\right)^2\right] = 337.8030\,\text{m/s}$$

Reflected Shock:



Reflected Normal Shock in a Stationary Coordinate System

Intermediate Step

Reflected Normal Shock in a Moving Coordinate System

Now from Eq.(5.10) with S replaced by $S_R + V$ and V replaced by $V + V_p$ (see the intermediate step), i.e.,

$$S_R + V = \left(\frac{\gamma+1}{4}\right)(V+V_p) + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)(V+V_p)\right]^2 + a_2^2}$$

$$S_R = -V + \left(\frac{\gamma+1}{4}\right)(V+V_p) + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)(V+V_p)\right]^2 + a_2^2}$$

$$= -337.803 + (0.6)(337.803+60) + \sqrt{[(0.6)397.803]^2 + 440.3552^2}$$

$$= 401.7597\,\text{m/s}$$

So $M_2 = \frac{S_R + V}{a_2} = \frac{401.7597 + 337.8030}{440.3552} = 1.6795$

From the normal shock relations $\frac{p_3}{p_2} = 3.1242$. This can be verified by using Eq.(5.24).

The pressure behind the reflected shock, which is also the pressure on the piston face is

$$= p_1\left(\frac{p_2}{p_1}\right)\left(\frac{p_3}{p_2}\right) = (50\text{ kPa})(4.0494)(3.1242) = 632.5568\text{kPa}$$

Problem 9. – For both γ = 7/5 and 5/3, determine the limits of the pressure ratio of a reflected normal shock, i.e., p_3/p_2, (a) for a strong incident shock, i.e., $p_2/p_1 \to \infty$, and (b) for a weak incident wave, , i.e., $p_2/p_1 \to 1$.

From Eq.(5.24)

$$\frac{p_3}{p_2} = \frac{\left(\frac{3\gamma - 1}{\gamma + 1}\right)\left(\frac{p_2}{p_1}\right) - \left(\frac{\gamma - 1}{\gamma + 1}\right)}{1 + \left(\frac{\gamma - 1}{\gamma + 1}\right)\left(\frac{p_2}{p_1}\right)}$$

(a) For the strong shock case since p_2/p_1 is infinite the ratio simply becomes

$$\frac{p_3}{p_2} \to \frac{\left(\frac{3\gamma - 1}{\gamma + 1}\right)\left(\frac{p_2}{p_1}\right)}{\left(\frac{\gamma - 1}{\gamma + 1}\right)\left(\frac{p_2}{p_1}\right)} = \frac{(3\gamma - 1)}{(\gamma - 1)}$$

Thus,

$\frac{p_3}{p_2} = 6$ for γ = 5/3 and

$\frac{p_3}{p_2} = 8$ for γ = 7/5

(b) For a weak shock, p_2/p_1 is very close to 1, thus

$$\frac{p_3}{p_2} = \frac{\left(\frac{3\gamma-1}{\gamma+1}\right)(1) - \left(\frac{\gamma-1}{\gamma+1}\right)}{1+\left(\frac{\gamma-1}{\gamma+1}\right)(1)} = \frac{(3\gamma-1)-(\gamma-1)}{(\gamma+1)+(\gamma-1)} = \frac{2\gamma}{2\gamma} = 1$$

Problem 10. – A shock tube is to be used to subject an object to momentary conditions of high pressure and temperature. To provide an adequate measuring time, the tube is to be made long enough so that a period of 100 ms is provided between the time of passage over the body of the initial shock and the time of passage of the shock reflected from the closed end of the tube. The initial pressure ratio across the diaphragm is 400 to 1, with the object located 3 m from the diaphragm. The initial temperature of the air ($\gamma = 1.4$, $R = 287$ J/kg·K) in the shock tube is 35°C. Determine a suitable length for the low-pressure end of the tube.

Incident Shock:

To begin we can calculate $a_1 = \sqrt{(1.4)(287)(308)} = 351.7874\,\text{m/s}$. For $p_4/p_1 = 400$ we find p_2/p_1 using the iterative procedure described in Example 5.6. From that calculation the shock pressure ratio is determined to be $p_2/p_1 = 9.2853$. Now from Eq,(4.12)

$$M_1 = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}\right)} = \sqrt{\frac{6}{7}\left(9.2853 + \frac{1}{6}\right)} = 2.8463$$

$$S_I = M_1 a_1 = (2.8463)(351.7874) = 1001.3087\,\text{m/s}$$

Now at M_1 we can also find the density ratio and temperature ratio across the shock. The first ratio will give the velocity behind the shock, V_2, and the second will produce a_2.

$$\frac{\rho_2}{\rho_1} = 3.7102, \quad \frac{T_2}{T_1} = 2.5026$$

Hence,

$$\frac{\rho_2}{\rho_1} = \frac{S_I}{S_I - V_2} = \frac{1001.3087}{1001.3087 - V_2} = 3.7102$$

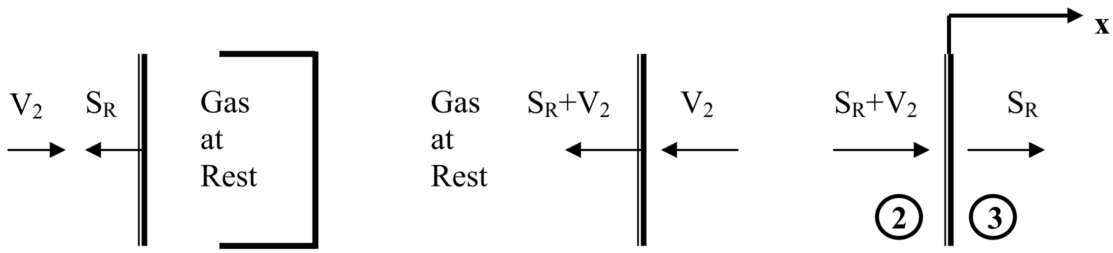
From which we find, $V_2 = 731.4287$ m/s. Also,

$$T_2 = \left(\frac{T_2}{T_1}\right) T_1 = 2.5026(308) = 770.8008\,\text{K}.$$

Thus,

$$a_2 = \sqrt{(1.4)(287)(770.8008)} = 556.5139\,\mathrm{m/s}$$

Reflected Shock:



Reflected Normal Shock in a Stationary Coordinate System

Intermediate Step

Reflected Normal Shock in a Moving Coordinate System

To compute the reflected shock speed use Eq.(5.10). However the speed S that appears in the expression must be replaced with $S_R + V_2$ and V is replaced with V_2 to agree with the current problem. Accordingly we may write:

$$S_R + V_2 = \left(\frac{\gamma+1}{4}\right)V_2 + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V_2\right]^2 + a_2^2}$$

Thus,

$$S_R = \left(\frac{\gamma-3}{4}\right)V_2 + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V_2\right]^2 + a_2^2}$$

$$= -(0.4)731.4287 + \sqrt{[(0.6)731.4287]^2 + 556.5139^2}$$

$$= 416.1622\,\mathrm{m/s}$$

$$\text{Time to test} = \frac{L-3}{1001.3087} + \frac{L-3}{416.1622} = 100 \times 10^{-3}\,\mathrm{s}$$

$$L = 3 + 29.3979 = 32.3979\ \mathrm{m}$$

Problem 11. – Air (γ = 1.4, R = 287 J/kg·K) is stored in a tube at 200 kPa and 300 K (Figure P.5.11). A diaphragm at the end of the tube separates the high-pressure air and the ambient, which has a pressure of 101 kPa. The diaphragm is suddenly ruptured, which causes expansion waves to move down the duct. Determine the time required for the first expansion wave to reach the closed end of the tube and the velocity of the air behind the expansion waves.

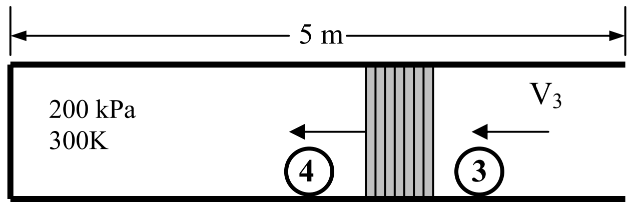


Figure P5.11

$$\text{Time} = \frac{L}{a_4} = \frac{5}{\sqrt{(1.4)(287)(300)}} = \frac{5}{347.1887} = 0.01440\,\text{s}$$

$$V_3 = \frac{2a_4}{\gamma_4 - 1}\left[1 - \left(\frac{p_3}{p_4}\right)^{\frac{\gamma_4 - 1}{2\gamma_4}}\right] = \frac{2(347.1887)}{0.4}\left[1 - \left(\frac{101}{200}\right)^{\frac{0.4}{2.8}}\right]$$

$$= 1735.9435(1 - 0.9070) = 161.4218\,\text{m/s}$$

Problem 12. – Write a computer program that will yield values of the diaphragm pressure ratio for given values of the shock pressure ratio for a shock tube with helium ($\gamma = 5/3$) with the same temperature on both sides of the diaphragm. Determine values of diaphragm pressure ratio for shock pressure ratios from 1.0 to 5.0, using increments of 0.2.

From Eq.(5.36), with $\gamma_4 = \gamma_1$, $a_4 = a_1$, and $p = p_2/p_1$

$$\frac{p_4}{p_1} = \frac{p_2}{p_1}\left[1 - \frac{(\gamma_4 - 1)\left(\frac{a_1}{a_4}\right)\left(\frac{p_2}{p_1} - 1\right)}{\sqrt{2\gamma_1}\sqrt{2\gamma_1 + (\gamma_1 + 1)\left(\frac{p_2}{p_1} - 1\right)}}\right]^{-\frac{2\gamma_4}{\gamma_4 - 1}} = \frac{p}{\left[1 - \frac{(\gamma - 1)(p - 1)}{\sqrt{2\gamma}\sqrt{2\gamma + (\gamma + 1)(p - 1)}}\right]^{\frac{2\gamma}{\gamma - 1}}}$$

and with $\gamma = 5/3$ this becomes

$$\frac{p_4}{p_1} = \frac{p}{\left[1 - \frac{0.2(p-1)}{\sqrt{1+0.8(p-1)}}\right]^5}$$

The spreadsheet program developed for this problem is as follows:

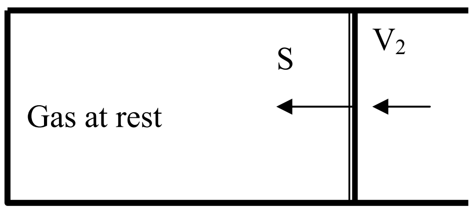
p2/p1	p4/p1
1	=H4/(1-((0.2)*(H4-1)/(SQRT(1+0.8*(H4-1)))))^5
=H4+0.2	=H5/(1-((0.2)*(H5-1)/(SQRT(1+0.8*(H5-1)))))^5
=H5+0.2	=H6/(1-((0.2)*(H6-1)/(SQRT(1+0.8*(H6-1)))))^5

The results are

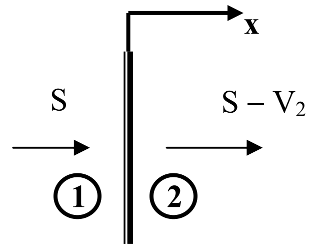
p_2/p_1	p_4/p_1		p_2/p_1	p_4/p_1
1.00	1.00000		3.00	12.48052
1.20	1.44998		3.20	14.90276
1.40	2.00841		3.40	17.68189
1.60	2.68922		3.60	20.86205
1.80	3.50817		3.80	24.49276
2.00	4.48295		4.00	28.62954
2.20	5.63343		4.20	33.33473
2.40	6.98190		4.40	38.67827
2.60	8.55335		4.60	44.73876
2.80	10.37576		4.80	51.60453
3.00	12.48052		5.00	59.37489

Problem 13. – A circular tube of length 1.5 m is evacuated to a pressure of 2.5 kPa, with the ambient pressure at 101 kPa. A diaphragm at the end of the tube is ruptured, which causes a normal shock to move down the tube. Determine the velocity of the initial shock that moves down the tube, the velocity and Mach number of the air (γ = 1.4, R = 287 J/kg·K) behind the shock, and the velocity of the shock that reflects from the closed end. Initial air temperature before diaphragm rupture is 300 K. A test object is located midway along the tube. Determine the time that this object is subjected to the pressure and temperature conditions behind the initial shock (before arrival of the reflected shock). Find the static pressure and temperature behind the initial shock.

Initial Shock: Fix the shock by redefining the coordinate system



Incident Normal Shock in a Stationary Coordinate System



Incident Normal Shock in a Moving Coordinate System

To begin we can calculate $a_1 = \sqrt{(1.4)(287)(300)} = 347.1887\ \text{m/s}$. Since we are given p_2 = 101 kPa and p_1 = 2.5 kPa, then

$$\frac{p_2}{p_1} = \frac{101}{2.5} = 40.40$$

From Eq,(4.12)

$$M_1 = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}\right)} = \sqrt{\frac{6}{7}\left(40.4 + \frac{1}{6}\right)} = 5.8967$$

$$S = M_1 a_1 = (5.8967)347.1887 = 2047.2781\text{m/s}$$

Now at M_1 we can also find the density ratio and temperature ratio across the shock. The first ratio will give the velocity behind the shock, V_2, and the second will produce a_2.

$$\frac{\rho_2}{\rho_1} = 5.2457, \quad \frac{T_2}{T_1} = 7.7016$$

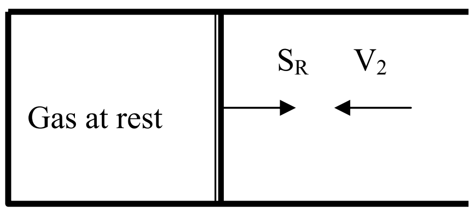
Hence,

$$\frac{\rho_2}{\rho_1} = \frac{S}{S - V_2} = \frac{2047.2781}{2047.2781 - V_2} = 5.2457$$

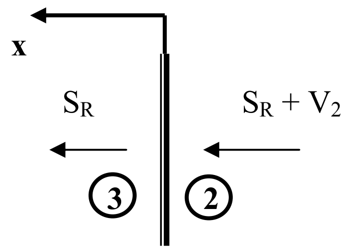
From which we find, V_2 = 1,657.0007 m/s (the velocity of the air behind shock). Also, $T_2 = \left(\frac{T_2}{T_1}\right)T_1 = 7.7016(300) = 2{,}310.48\ \text{K}$ (the temperature of the air behind the initial shock). Thus, $a_2 = \sqrt{(1.4)(287)(2310.48)} = 963.5097\ \text{m/s}$

Mach number of the air behind the initial shock = $\frac{V_2}{a_2} = \frac{1{,}657.0007}{963.5097} = 1.7198$. The pressure behind the shock is p_2 = 101 kPa.

Reflected Shock: Define a moving coordinate system for the reflected wave as usual.



Reflected Normal Shock in a Stationary Coordinate System



Reflected Normal Shock in a Moving Coordinate System

Replace S and V in Eq.(5.10) with $S_R + V_2$ and V_2, respectively and rewrite the expression as

$$S_R = \left(\frac{\gamma-3}{4}\right)V_2 + \sqrt{\left[\left(\frac{\gamma+1}{4}\right)V_2\right]^2 + a_2^2}$$

$$= -(0.4)1657.0007 + \sqrt{[(0.6)1657.0007]^2 + 963.5097^2}$$

$$= 721.6799 \text{ m/s}$$

$$\Delta t = \frac{0.75}{2047.2781} + \frac{0.75}{721.6799} = 3.663\times10^{-4} + 10.392\times10^{-4} = 0.001406\,\text{s}$$

Problem 14. – The pressure ratio across the diaphragm in a shock tube is set at 10. The diaphragm is ruptured. Determine the velocity of the initial normal shock, the Mach number of the gas behind the shock, and the static pressure and temperature behind the shock for air (γ = 1.4, R = 287 J/kg·K) as the working fluid and for helium (γ = 5/3, R = 2.077 kJ/kg·K).as the working fluid. Assume the initial temperature on each side of the diaphragm to be 25°C and the initial pressure in the low-pressure end to be 25 kPa.

Air:

$$a_1 = \sqrt{\gamma RT_1} = \sqrt{(1.4)(287)(298)} = 346.0295\,\text{m/s}$$

Using the iterative procedure described in Example problem 5.6 for $p_4/p_1 = 10$ and $\gamma = 1.4$, we find :

shock pressure ratio: $\dfrac{p_2}{p_1} = 2.8482$ from which

$$M_1 = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}\right)} = \sqrt{\frac{6}{7}\left(2.8482 + \frac{1}{6}\right)} = 1.6075$$

$$S = M_1 a_1 = (1.6075)(346.0295) = 556.2548\,\text{m/s}$$

Now at $M_1 = 1.6075$ from the shock relations

$$\frac{T_2}{T_1} = 1.3932$$

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 = 1.3932(298) = 415.1736\,\text{K}$$

$$a_2 = \sqrt{\gamma RT_2} = \sqrt{(1.4)(287)(415.1736)} = 408.4321\,\text{m/s}$$

$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 = 2.8482(25) = 71.2050\,\text{kPa}$$

From Eq.(5.9)

$$V_2 = \frac{2S}{\gamma+1}\left[1 - \left(\frac{a_1}{S}\right)^2\right] = \frac{2(556.2548)}{2.4}\left[1 - \left(\frac{346.0295}{556.2548}\right)^2\right] = 284.1668\,\text{m/s}$$

Finally the Mach number behind the shock is

$$M = \frac{V_2}{a_2} = \frac{284.1668}{408.4321} = 0.6958$$

Helium:

$$a_1 = \sqrt{\gamma RT_1} = \sqrt{(1.667)(2077)(298)} = 1{,}015.6656\,\text{m/s}$$

Using the iterative procedure described in Example problem 5.6 for $p_4/p_1 = 10$ and $\gamma = 5/3$, we find :

$$\text{shock pressure ratio } \frac{p_2}{p_1} = 2.7611 \text{ from which}$$

$$M_1 = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} + \frac{\gamma-1}{\gamma+1}\right)} = \sqrt{\frac{8}{10}\left(2.7611 + \frac{1}{4}\right)} = 1.5521$$

$$S = M_1 a_1 = (1.5521)(1{,}015.6656) = 1{,}576.3646\,\mathrm{m/s}$$

Now at $M_1 = 1.5521$ from the shock relations

$$\frac{T_2}{T_1} = 1.5500$$

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 = (1.5500)(298) = 461.9000\mathrm{K}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{(1.6667)(2077)(461.9)} = 1{,}264.4935\,\mathrm{m/s}$$

$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 = (2.7611)(25) = 69.0275\,\mathrm{kPa}$$

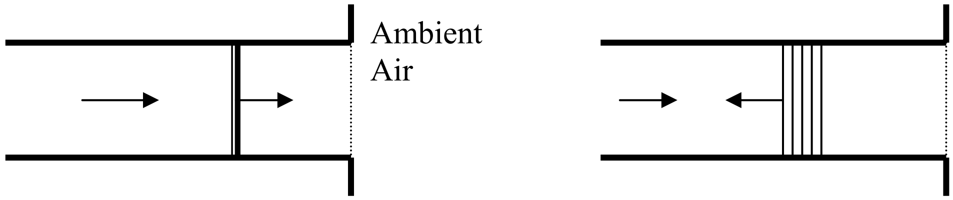
From Eq.(5.9)

$$V_2 = \frac{2S}{\gamma+1}\left[1 - \left(\frac{a_1}{S}\right)^2\right] = \frac{2(1{,}576.3646)}{2.6667}\left[1 - \left(\frac{1{,}015.6656}{1{,}576.3646}\right)^2\right] = 691.4717\,\mathrm{m/s}$$

Finally the Mach number behind the shock is

$$M = \frac{V_2}{a_2} = \frac{691.4717}{1{,}264.4935} = 0.5468$$

Problem 15. – A normal shock moves down an open-ended tube with a velocity of 1,000 m/s (Figure P5.15). The ambient air (γ = 1.4, R = 287 J/kg·K) pressure and temperature are 101 kPa and 25°C, respectively. Determine the velocity of the first and last expansion waves that move down the tube after reflection of the shock from the open end.

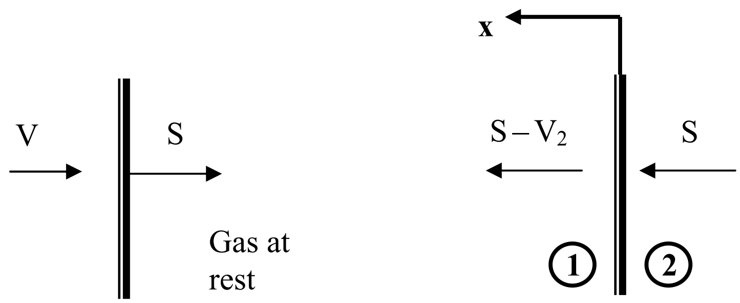


Incident Normal Shock on Open End of Tube

Reflected Expansion Waves

Figure P5.15

Shock: Fix the moving shock by defining a moving coordinate system



Moving Shock in a Stationary Coordinate System

Incident Normal Shock in a Moving Coordinate System

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)(287)(298)} = 346.0295\,\text{m/s}$$

$$M_1 = \frac{S}{a_1} = \frac{1000}{346.0295} = 2.8899$$

From the shock relations at this Mach number we obtain

$$\frac{p_2}{p_1} = 9.5768,\ \frac{\rho_2}{\rho_1} = 3.7531,\ \frac{T_2}{T_1} = 2.5517$$

$$\frac{\rho_2}{\rho_1} = \frac{S}{S - V_2} = \frac{1000}{1000 - V_2} = 3.7531$$

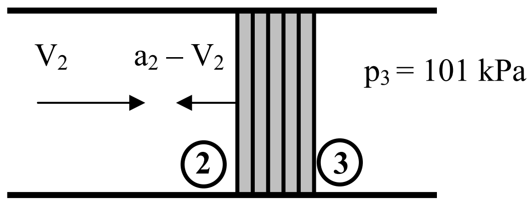
$V_2 = 733.5536$ m/s

$$T_2 = (2.5517)(298) = 760.4066\ \text{K}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{(1.4)(287)(760.4066)} = 552.7489\ \text{m/s}$$

$$p_2 = 9.5768(101) = 967.2568\ \text{kPa}$$

Expansion Waves:



Moving Expansion Waves

$$\text{Velocity of first wave } = a_2 - V_2 = 552.7489 - 733.5536 = -180.8047\ \text{m/s}$$

The minus sign means that it is moving to the right, i.e., because V_2 exceeds the speed of sound, the disturbance is unable to move upstream. Because the flow in the expansion fan is isentropic

$$\frac{p_3}{p_2} = \left(\frac{a_3}{a_2}\right)^{2\gamma/(\gamma-1)}$$

$$\frac{a_3}{a_2} = \left(\frac{p_3}{p_2}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{101}{967.2568}\right)^{\frac{0.4}{2.8}} = \left(\frac{101}{967.2568}\right)^{\frac{1}{7}} = 0.7241$$

$$a_3 = a_2(0.7241) = (552.7489)(0.7241) = 400.2705\ \text{m/s}$$

Now for a left running wave,

$$V_2 + \frac{2}{\gamma - 1} a_2 = V_3 + \frac{2}{\gamma - 1} a_3$$

So

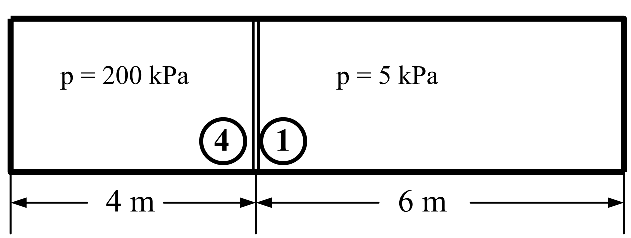
$$V_3 = V_2 + \left(\frac{2}{\gamma - 1}\right)(a_2 - a_3) = 733.5536 + 5(552.7489 - 400.2705)$$

$$= 1495.9458\,\text{m/s}$$

$$\text{Velocity of last wave } = a_3 - V_3 = 400.2705 - 1495.9458 = -1095.6753\,\text{m/s}$$

Problem 16. – A shock tube is 10 m long with a 30-cm diameter. The high-pressure section is 4 m long and contains air (γ = 1.4, R = 287 J/kg·K) at 200 kPa; the low-pressure section is 6 m long and contains air at 5 kPa. A test object is placed in the low-pressure section, 3 m from the diaphragm. Both sections initially contain air at 25°C. The diaphragm is suddenly ruptured, which causes a shock to move into the low-pressure section. Determine the following:

(a) Shock velocity
(b) Contact surface velocity
(c) Mach number of air behind shock
(d) Time between passage of normal shock and contact surface over test object
(e) Reflected shock velocity
(f) Sketch a x-t diagram showing the initial shock, reflected shock, and contact surface as functions of time.



(a) For a diaphragm pressure ratio = 40, we may use the method described in Example 5.6 to find that the shock pressure ratio is,

$$\frac{p_2}{p_1} = 4.7726$$

With this pressure ratio and the speed of sound in Zone 1, (a_1 = 346.0295 m/s), we can find the shock speed from Eq. (5.8)

$$S = 711.9821\,\text{m/s}$$

(b) $$V_2 = \frac{2S}{\gamma+1}\left(1-\frac{a_1^2}{S^2}\right) = \frac{2(711.9821)}{2.4}\left(1-\frac{1}{(2.0576)^2}\right) = 453.1740\ \text{m/s}$$

$$T_2 = \left(\frac{T_2}{T_1}\right)T_1 = 1.7348(298) = 516.9704\ \text{K}$$

(c)

$$a_2 = \sqrt{(1.4)(287)(516.9704)} = 455.7480\ \text{m/s}$$

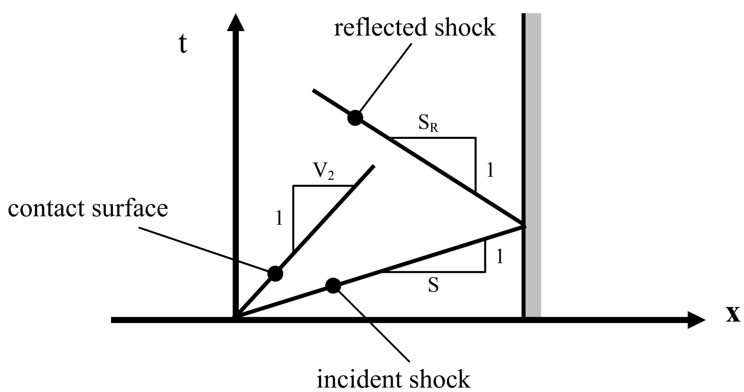
$$M_2 = \frac{V_2}{a_2} = \frac{453.1740}{455.7480} = 0.9944$$

(d) $$\Delta t = \frac{L}{V_2} - \frac{L}{S} = \frac{3}{453.174} - \frac{3}{711.9821} = .006620 - .004214 = 0.002406\ \text{s}$$

(e) From Eq.(5.21)

$$S_R = -V_2 + \frac{a_2^2}{S - V_2} = -453.174 + \frac{455.748^2}{711.9821 - 453.174} = 349.3752\ \text{m/s}$$

(f)



Chapter Six

OBLIQUE SHOCK WAVES

Problem 1. – Uniform airflow ($\gamma = 1.4$, R = 287 J/kg·K) at Mach 3 passes into a concave corner of angle 15°, as shown in Figure P6.1. The pressure and temperature in the supersonic flow are, respectively, 72 kPa and 290 K. Determine the tangential and normal components of velocity and Mach number upstream and downstream of the wave for the weak shock solution. Also find the static and stagnation pressure ratios across the wave. How great would the corner angle have to be before the shock would detach from the corner?

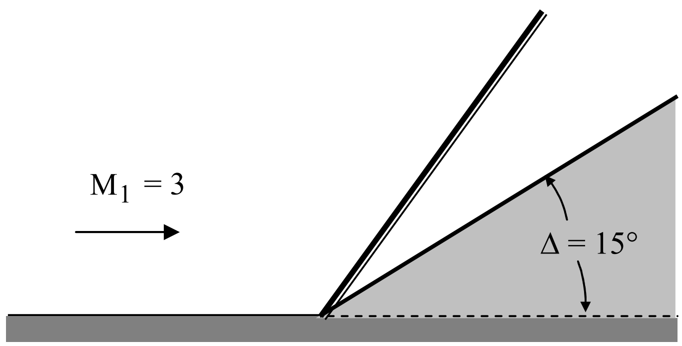


Figure P6.1

For $M_1 = 3.0$, $\delta = 15°$: Collar's method (refer to Example 6.2 for details of the method) is used to find the shock angle. The following provides the iteration details.

Collar's Method				
A	**B**	**C**	**B - AC**	**1rst guess**
8.0000	26.0447	3.1618	0.7503	2.8284

Newton-Raphson Method				
iteration	$\mathbf{x_{old}}$	**f**	**fprime**	$\mathbf{x_{new}}$
1	2.8284	26.0447	33.8858	2.0598
2	2.0598	6.4264	17.7542	1.6979
3	1.6979	1.1765	11.3848	1.5945
4	1.5945	0.0871	9.7107	1.5856
5	1.5856	6.3779E-04	9.5684	1.5855
6	1.5855	3.5182E-08	9.5674	1.5855
7	1.5855	0.0000E+00	9.5674	1.5855

	$\cot\theta$	$\tan\theta$	angle (deg)
weak	1.5855	0.6307	32.24
strong	0.0977	10.2387	84.42
neg root	-4.8450	-0.2064	-11.66

So for the weak solution, the shock angle is 32.24°

$$M_{n1} = M_1 \sin\theta = 3\sin(32.24°) = 1.6004$$

From the shock tables at this Mach number M_{n2} = 0.6683, p_2/p_1 = 2.8215, T_2/T_1 = 1.3882 and p_{o2}/p_{o1} = 0.8950. From Eq.(6.9b)

$$M_2 = \frac{M_{2n}}{\sin(\theta-\delta)} = \frac{0.6683}{\sin(32.24-15)} = 2.2549$$

Too be sure, these could also be computed from the oblique shock relations of this Chapter [Eqs.(6.10), (6.12), (6.13) and (6.17)].

From the isentropic tables at M_1 = 3.0, T/T_o = 0.3571

$$T_{o2} = T_{o1} = \frac{290}{0.3571} = 812.1\,\text{K}$$

Also,

$$T_2 = (1.3883)T_1 = (1.3883)290 = 402.6\text{K}$$

So the speeds of sound may be computed as

$$a_1 = \sqrt{\gamma R T_1} = 20.05\sqrt{290} = 341.4\text{m/s}$$

$$a_2 = \sqrt{\gamma R T_2} = 20.05\sqrt{402.6} = 402.3\text{m/s}$$

And the normal velocity components are

$$V_{n1} = M_{n1}a_1 = 1.6004(341.4) = 545.9\text{ m/s}$$

$$V_{n2} = M_{n2}a_2 = 0.6683(402.3) = 268.9\text{ m/s}$$

Also,

$$V_1 = M_1 a_1 = 3.0(341.4) = 1024.2\text{ m/s}$$

$$V_2 = M_2 a_2 = 2.2549(402.3) = 907.1\text{ m/s}$$

The tangential velocity component can be computed from either Eq.(6.6a) or (6.7a)

$$V_t = V_1 \cos\theta = 1024.2\cos(32.24°) = 866.3 \text{ m/s}$$

$$V_t = V_2 \cos(\theta - \delta) = 907.1\cos(32.24 - 15) = 866.3 \text{ m/s}$$

$$M_{t1} = \frac{V_t}{a_1} = \frac{866.3}{341.4} = 2.54$$

$$M_{t2} = \frac{V_t}{a_2} = \frac{866.3}{402.3} = 2.15$$

From Table 6.4 for $\gamma = 1.4$ and $M_1 = 3.0$, δ_{max} is found to be 34.07°.

Problem 2. – In a helium ($\gamma = 5/3$) wind tunnel, flow at Mach 4.0 passes over a wedge of unknown half-angle aligned symmetrically with the flow. An oblique shock is observed attached to the wedge, making an angle of 30° with the flow direction. Determine the half-angle of the wedge and the ratios of stagnation pressure and stagnation temperature across the wave.

Method 1: Use of normal shock tables.

$$M_{n1} = 4\sin(30°) = 2.0$$

Using this value we can enter the normal shock table at a $\gamma = 5/3$ to find

$$\frac{T_2}{T_1} = 2.0781$$

$$\frac{p_{o2}}{p_{o1}} = 0.763$$

$$\frac{T_{o2}}{T_{o1}} = 1.0$$

$$M_{n2} = 0.607$$

Entering the isentropic flow table at $M_1 = 4$ we find that

$$\frac{T_1}{T_{o1}} = 0.1579$$

Thus,

$$\frac{T_2}{T_{o2}} = \left(\frac{T_2}{T_1}\right)\left(\frac{T_1}{T_{o1}}\right) = (2.0781)(0.1579) = 0.3281$$

Entering the isentropic flow tables with this temperature ratio provides the downstream Mach number

$$M_2 = 2.479$$

$$\sin(\theta - \delta) = \frac{M_{n2}}{M_2} = \frac{0.607}{2.479} = 0.2449$$

$$\theta - \delta = 30 - \delta = 14.17°$$

Therefore, the deflection angle δ = 15.83°.

Method 2: Oblique shock equations

γ	M_1	θ	δ	p_2/p_1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}	M_2
1.6667	*4.0000*	*30.0000*	15.8241	4.7500	2.2857	2.0781	0.7630	2.4785

Problem 3. – A wedge is to be used as an instrument to determine the Mach number of a supersonic airstream (γ = 1.4); that is, with the wedge axis aligned to the flow, the wave angle of the attached oblique shock is measured; this permits a determination of the incident Mach number. If the total included angle of such a wedge is 45°, give the Mach number range over which such an instrument would be effective.

Refer to Example 6.3. That example concerned the prediction of the minimum upstream Mach number to produce an attached oblique shock. This is a similar problem; only here, the half angle Δ = 45/2 = 22.5°. So for a deflection angle δ of 22.5° and γ = 1.4, computations following Example 6.3 are as follows:

Iteration	θ (deg)	θ(rad)	f(θ)	df/dθ	θnew	θ (deg)	$1/M_1^2$	M_1
1	45.00000	0.78540	0.85858	-1.41421	1.39250	79.78466	0.13234	2.74886
2	79.78466	1.39250	-0.64460	-1.72985	1.01987	58.43444	0.24273	2.02971
3	58.43444	1.01987	3.2111E-01	-2.88069	1.13134	64.82124	0.25692	1.97287
4	64.82124	1.13134	-6.5888E-03	-2.92848	1.12909	64.69233	0.25693	1.97284
5	64.69233	1.12909	3.5501E-06	-2.93161	1.12910	64.69240	0.25693	1.97284
6	64.69240	1.12910	9.9343E-13	-2.93161	1.12910	64.69240	0.25693	1.97284
7	64.69240	1.12910	0.0000E+00	-2.93161	1.12910	64.69240	0.25693	1.97284

Thus as long as $M_1 \geq 1.97284$ the shock will remain attached to the wedge.

Problem 4. – The leading edge of a supersonic wing is wedge shaped, with a total included angle of 10° (Figure P6.4). If the wing is flying at zero angle of attack, determine the lift and drag force on the wing per meter of span. Repeat for an angle of attack of 3°. Assume the wing is traveling at Mach 2.5.

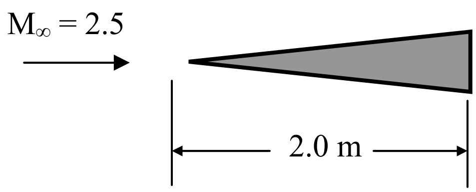
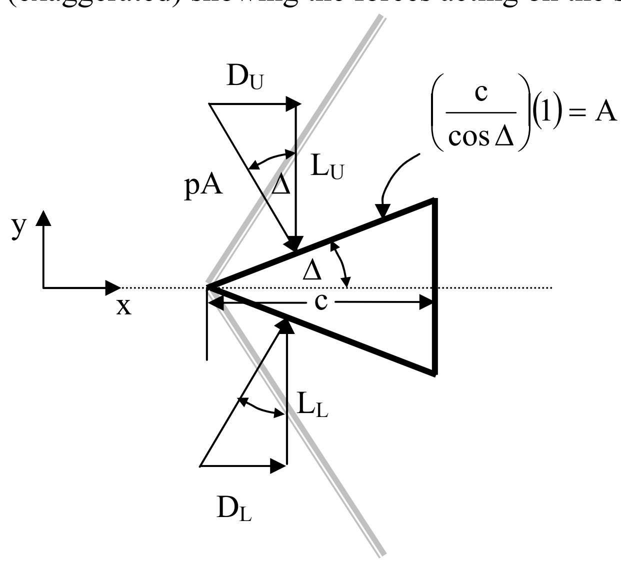


Figure P6.4

Case I: *Zero angle of attack*:

First draw a figure (exaggerated) showing the forces acting on the surface:



10º

10º

10º

$$\text{Lift} = \sum F_y = pA\sin\Delta - pA\sin\Delta = 0$$

For $M_1 = 2.5,\ \ \delta = 5°$ using Collar's method we find θ = 27.4227º. With this shock angle and the Mach number we can determine the pressure ratio across the shock to be

$$\frac{p_2}{p_1} = 1.3799, \quad p_{surface} = 1.3799\, p_\infty$$

$$\text{Drag force} = \sum F_x = pA\sin\Delta + pA\sin\Delta = 2pA\sin\Delta$$

Thus,

$$\text{Drag} = 2(1.3799p_\infty)\left(\frac{2}{\cos\ 5°}\right)\ \sin\ 5° = 0.4829p_\infty$$

Case II: *Angle of attack = 3°:*

For upper surface: $\delta = 2°$ and $M_1 = 2.5$, we find $\theta = 25.0496°$. So that

$$M_{n1} = M_1 \sin\theta = 2.5 \sin 25.0496 = 1.0585$$

Using the normal shock relations we obtain

$$\frac{p_U}{p_\infty} = 1.1405$$

For lower surface: $\delta = 2°$ and $M_1 = 2.5$, we find $\theta = 30.0053°$. So that

$$M_{n1} = 2.5 \sin 30.0053 = 1.2502$$

$$\frac{p_L}{p_\infty} = 1.6568$$

$$\text{Drag} = p_L\ A \sin\ 8° + p_U\ A \sin\ 2°$$

$$= (1.6568p_\infty)\frac{2}{\cos 5°}\sin\ 8° + (1.1405p_\infty)\frac{2}{\cos 5°}\sin\ 2°$$

$$= 0.4629p_\infty + 0.0799p_\infty$$

$$= 0.5428p_\infty \text{with } p_\infty \text{ in kPa,} \quad \text{Drag in kN/m}$$

$$\text{Lift} = p_L \frac{2}{\cos 5°}\cos\ 8° - p_U \frac{2}{\cos 5°}\cos\ 2°$$

$$= (1.6568p_\infty)\frac{2}{\cos 5°}\cos\ 8° - (1.1405p_\infty)\frac{2}{\cos 5°}\cos\ 2°$$

$$= 3.2939p_\infty - 2.2883p_\infty$$

$$= 1.0056p_\infty \text{with } p_\infty \text{ in kPa,} \quad \text{Lift in kN/m}$$

Problem 5. – An oblique shock wave is incident on a solid boundary, as shown in Figure P6.5. The boundary is to be turned through such an angle that there will be no reflected wave. Determine the angle β.

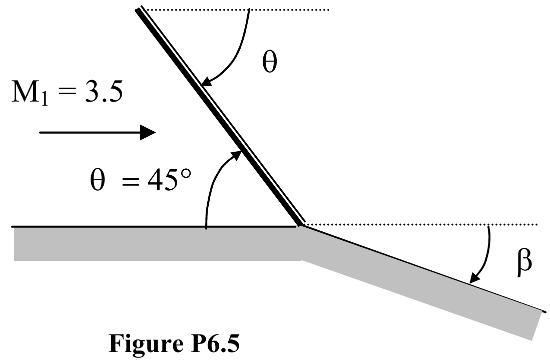


Figure P6.5

The given information M1 = 3.5, γ = 1.4 and θ = 45°is inserted into Eq.(6.18)

$$\tan\delta = \cot\theta \frac{M_1^2 \sin^2\theta - 1}{\frac{\gamma+1}{2}M_1^2 - \left(M_1^2 \sin^2\theta - 1\right)}$$

And we find that δ = 28.1578°. If the wall is turned through the same angle then there will be no need of a reflected oblique shock to turn the flow further.

Problem 6. – Explain in physical terms why the angle of incidence and the angle of reflection of a reflected oblique shock are not equal.

Whereas each shock turns the flow through the same angle, the shocks are of different strengths so the wave angles must be different.

Problem 7. –A converging-diverging nozzle is designed to provide flow at Mach 2.0. With the nozzle exhausting to a back pressure of 80 kPa, however, and a reservoir pressure of 280 kPa, the nozzle is overexpanded, with oblique shocks at the exit (Figure P6.7). Determine the flow direction and flow Mach number in region *R* with air the working fluid.

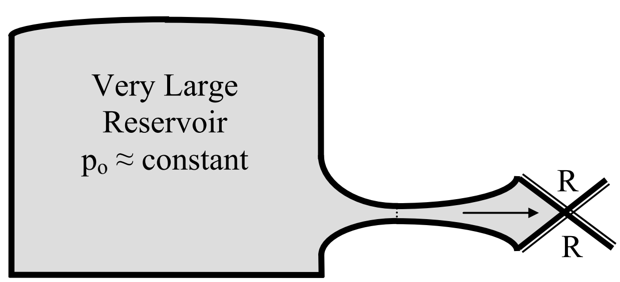


Figure P6.7

At the exit plane,

$$M_e = M_1 = 2.0,\ \ p_e = p_1 = \left(\frac{p_1}{p_{o1}}\right)p_o = (0.1278)280 = 35.7840 \text{ kPa}$$

Across shock, $\dfrac{p_2}{p_1} = \dfrac{p_b}{p_e} = \dfrac{80}{35.7840} = 2.2356$

Entering the normal shock tables at this pressure ratio we find, $M_{n1} = 1.4350$

$$M_{n1} = M_1 \sin\theta \ \text{ so } \ \theta = \sin^{-1}\left(\frac{1.4350}{2}\right) = 45.8485°$$

Now at M_{n1}

$. \dfrac{T_2}{T_1} = 1.2774$ Also at $M_1 = 2.0$ $\dfrac{T_1}{T_{o1}} = 0.5556$.

Therefore, $\dfrac{T_2}{T_{o2}} = \left(\dfrac{T_2}{T_1}\right)\left(\dfrac{T_1}{T_{o1}}\right)\left(\dfrac{T_{o1}}{T_{o2}}\right) = (1.2774)(0.5556)(1.0) = 0.7097$

At this static to total temperature ratio we can find $M_2 = 1.4301$

$$\sin(\theta - \delta) = \frac{M_{n2}}{M_2} = \frac{0.7254}{1.4301} = 0.5072$$

$$\delta = \theta - \sin^{-1}(0.5072) = 45.8485 - 30.4800 = 15.3685° \text{ in R}$$

Problem 8. –(a) Oblique shock waves appear at the exit of a supersonic nozzle, as shown in Figure P6.8. Air is the working fluid. If the nozzle back pressure is 101 kPa, determine the nozzle inlet stagnation pressure. The stagnation temperature of the flow is 500 K. Nozzle throat area is 50 cm^2, and nozzle exit area is 120 cm^2. (b) Find the velocity at the nozzle exit plane. (c) Find the mass flow rate through the nozzle.

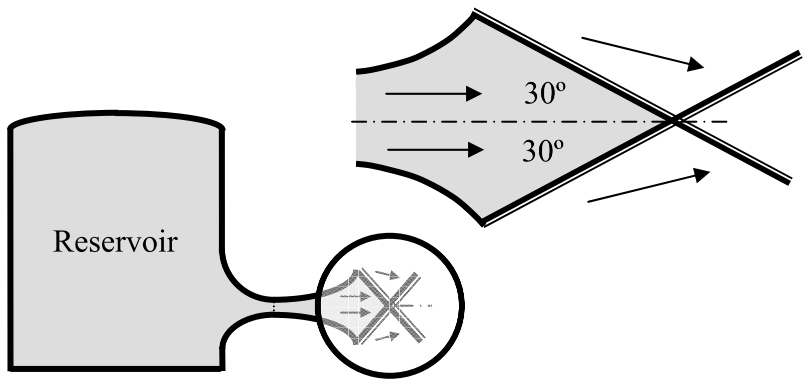


Figure P6.8

(a) $$\frac{A_{exit}}{A_{throat}} = \frac{A_e}{A^*} = \frac{120}{50} = 2.4$$

At this area ratio, the Mach number at the exit plane is $M_e = M_1 = 2.3986$, which with a shock angle $\theta = 30^o$, when used in Eq.(6.18), the deflection equation gives $\delta = 6.6970^o$ and $\frac{p_2}{p_1} = 1.5114$. Thus, $p_1 = \frac{101}{1.5114} = 66.8255$ kPa. Now at $M_1 =$ 2.3986, $\frac{p_1}{p_{o1}} = 0.0685$. Therefore,

$$p_{o1} = \frac{66.8255}{0.0685} = 975.5542 \text{ kPa}$$

(b) $$\frac{T_e}{T_o} = 0.4650 \quad \text{Thus, } T_e = \left(\frac{T_e}{T_o}\right)T_o = (0.4650)500 = 232.5000\text{K}$$

$$a_e = 20.05\sqrt{T_e} = 305.7214 \text{ m/s}$$

$$V_e = M_e a_e = 2.3986(305.7214) = 733.3034 \text{ m/s}$$

(c) $$\dot{m} = (\rho AV)_{throat}$$

$$\dot{m} = \left[\frac{(p_t/p_{o1})p_{o1}}{R(T_t/T_o)T_o}\right]A_tM_ta_t$$

$$= \frac{(0.5283)(975.5542)}{(0.287)(0.8333)500}\left(50\times10^{-4}\right)(1.0)20.05\sqrt{(0.8333)500}$$

$$= \left(4.3100\text{ kg/m}^3\right)\left(50\times10^{-4}\text{ m}^2\right)(409.2607\text{ m/s})$$

$$= 8.8196\text{ kg/s}$$

Problem 9. – A supersonic flow leaves a two-dimensional nozzle in parallel, horizontal flow (region A) with a Mach number of 2.6 and static pressure (in region A) of 50 kPa. The pressure of the atmosphere into which the jet discharges is 101 kPa. Find the pressures in regions B and C of Figure P6.9.

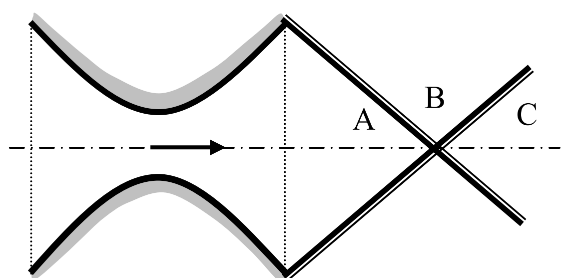


Figure P6.9

$p_2 = 101$ kPa, $\dfrac{p_2}{p_1} = \dfrac{101}{50} = 2.0200$. At this pressure ratio we can find the normal component to the shock, $M_{nA} = 1.3690$. Thus, at $M_A = 2.6$ and M_{nA}, we can find the shock wave angle,

$$\theta = \sin^{-1}\left(\frac{M_{nA}}{M_A}\right) = \sin^{-1}(0.5265) = 31.7719°$$

$M_1 = 2.6$, $\theta = 31.7719° \rightarrow \delta = 11.0346°$ So second shock must turn flow back by 11.0346°. With the flow angles and M_A, the temperature ratio across the shock is found to be

$$\frac{T_B}{T_A} = 1.2348,\quad \frac{T_B}{T_o} = \left(\frac{T_B}{T_A}\right)\left(\frac{T_A}{T_1}\right)\left(\frac{T_1}{T_o}\right) = (1.2348)(1.0)(0.4252) = 0.5250$$

Using this value with the isentropic flow relations gives

$M_B = 2.1269$

For $M = 2.1269, \quad \delta = 11.0346° \rightarrow \theta = 38.0563°$

$$\frac{p_C}{p_B} = 1.8388 \text{ so } p_C = (1.8388)101 = 185.7188 \text{ kPa}$$

Problem 10. – For the two-dimensional diffuser shown in Figure P6.10, find V_i and p_{oi},

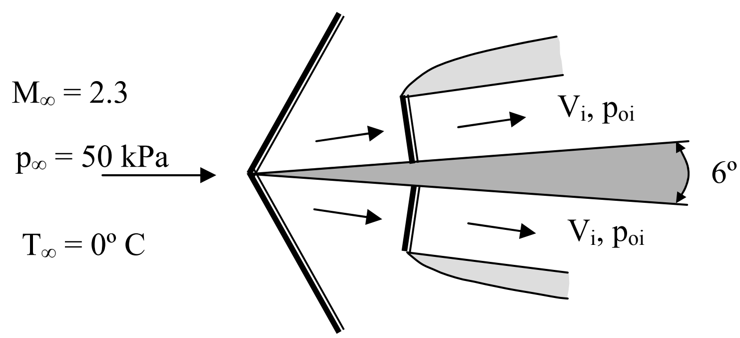


Figure P6.10

For an oblique shock with $\delta = 3°, \quad M_\infty = 2.3$ we can determine the shock wave angle using Collar's method: $\theta = 28.0886°$. Also at the freestream Mach number, $T_\infty/T_{o\infty} = 0.4859$ and $p_\infty/p_{o\infty} = 0.0800$.

$$p_{o\infty} = \frac{50}{.0800} = 625.0000 \text{ kPa}$$

There is enough information to determine the Mach number downstream of the shock $M_2 = 2.1823$ as well as several other ratios, viz., $\frac{T_2}{T_1} = 1.0540$ and $\frac{p_{o2}}{p_{o1}} = 0.9994$. Across the normal shock, at $M_2 = 2.1823$, $\frac{p_{o3}}{p_{o2}} = 0.6362$ and $M_3 = M_i = 0.5495$

$$p_{o1} = \left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right)\left(\frac{p_{o1}}{p_{o\infty}}\right)p_{o\infty} = (0.6362)(.9994)(1.0)625.0000 = 397.3864 \text{ kPa}$$

Now at $M_i = 0.5495, \quad \frac{T_i}{To} = 0.9430$

$$T_i = \left(\frac{T_i}{T_o}\right)\left(\frac{T_o}{T_\infty}\right)T_\infty = (0.9430)\left(\frac{1}{0.4859}\right)273 = 529.8189\ \text{K}$$

$$a_i = 20.05\sqrt{T_i} = 461.5068\text{m/s}$$

$$V_i = M_i a_i = (0.5495)461.5068 = 253.5980\ \text{m/s}$$

Problem 11. – A two-dimensional supersonic inlet is to be designed to operate at Mach 2.4. Deceleration is to occur through a series of oblique shocks followed by a normal shock, as shown in Figure 6.12. Determine the loss of stagnation pressure for the cases of two, three, and four oblique shocks. Assume the wedge turning angles are each 6°.

Case I:

Two oblique shocks:

$$\left.\begin{array}{l} M_1 = 2.4 \\ \delta = 6° \end{array}\right\} \begin{array}{l} M_2 = 2.1589 \\ \dfrac{p_{o2}}{p_{o1}} = 0.9948 \end{array}$$

$$\left.\begin{array}{l} M_2 = 2.1589 \\ \delta = 6° \end{array}\right\} \begin{array}{l} M_3 = 1.9354 \\ \dfrac{p_{o3}}{p_{o2}} = 0.9959 \end{array}$$

Normal Shock

$$M_3 = 1.9354,\ \frac{p_{o4}}{p_{o3}} = 0.7510$$

$$\frac{p_{o4}}{p_{o1}} = \frac{p_{o4}}{p_{o3}}\frac{p_{o3}}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.7510)(0.9959)(0.9948) = 0.7440$$

Three oblique shocks:

$$\left.\begin{array}{l} M_3 = 1.9354 \\ \delta = 6° \end{array}\right\} \begin{array}{l} M_4 = 1.7240 \\ \dfrac{p_{o4}}{p_{o3}} = 0.9966 \end{array}$$

Normal Shock

$$M_3 = 1.7240,\ \frac{p_{o5}}{p_{o4}} = 0.8457$$

$$\frac{p_{o5}}{p_{o1}} = \frac{p_{o5}}{p_{o4}}\frac{p_{o4}}{p_{o3}}\frac{p_{o3}}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.8457)(0.9966)(0.9959)(0.9948) = 0.8350$$

Four oblique shocks:

$$\left.\begin{array}{l} M_4 = 1.7240 \\ \delta = 6° \end{array}\right\} \begin{array}{l} M_5 = 1.5184 \\ \dfrac{p_{o5}}{p_{o4}} = 0.9972 \end{array}$$

Normal Shock

$$M_5 = 1.5184,\ \frac{p_{o6}}{p_{o5}} = 0.9239$$

$$\frac{p_{o6}}{p_{o1}} = \frac{p_{o6}}{p_{o5}}\frac{p_{o5}}{p_{o4}}\frac{p_{o4}}{p_{o3}}\frac{p_{o3}}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.9239)(0.9972)(0.9966)(0.9959)(0.9948) = 0.9097$$

Problem 12. – Two oblique shocks intersect as shown in Figure P6.12. Determine the flow conditions after the intersection, with $\gamma = 1.4$.

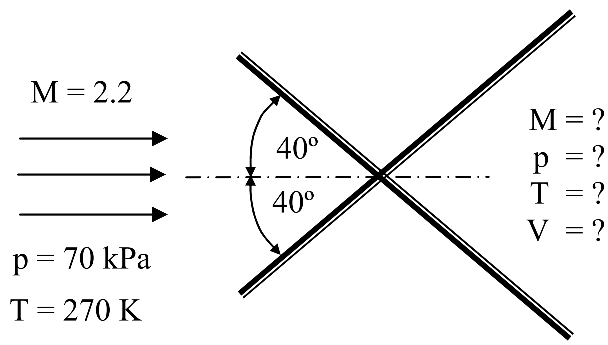


Figure P6.12

At $M_1 = 2.2, \quad \theta = 40°$

$$\delta = 13.9176°, \quad M_2 = 1.6691 \quad \frac{p_2}{p_1} = 2.1664 \quad \frac{T_2}{T_1} = 1.2638$$

At $M_2 = 1.6691, \quad \delta = 13.9176°$

$$M_3 = 1.1402, \quad \frac{p_3}{p_2} = 2.0253 \quad \frac{T_3}{T_2} = 1.2359$$

$$\frac{p_3}{p_1} = \frac{p_3}{p_2}\frac{p_2}{p_1} = (2.0253)(2.1664) = 4.3876$$

$$p_3 = \left(\frac{p_3}{p_1}\right) p_1 = (4.3876)70 = 307.1327 \text{ kPa}$$

$$T_3 = \left(\frac{T_3}{T_o}\right)\left(\frac{T_o}{T_1}\right) T_1 = (0.7936)\left(\frac{1}{0.5081}\right) 270 = 421.7123 \text{ K}$$

$$a_3 = 20.05\sqrt{T_3} = 411.7395 \text{ m/s}$$

$$V_3 = 1.1402(411.7395) = 469.4653 \text{ m/s}$$

Problem 13. – Show that the entropy increase across an oblique shock is given by, (Ref. 7)

$$\frac{\Delta s}{c_v} = \ln\left[\left(\frac{2\gamma}{\gamma+1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1) M_1^2 \sin^2\theta}\right)^{\gamma}\right]$$

From Eq.(6.13)

$$\frac{p_{o2}}{p_{o1}} = e^{-\frac{\Delta s}{R}} = \left[\frac{\frac{\gamma+1}{2} M_1^2 \sin^2\theta}{1 + \frac{\gamma-1}{2} M_1^2 \sin^2\theta}\right]^{\gamma/(\gamma-1)} \left[\frac{1}{\frac{2\gamma}{\gamma+1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1}}\right]^{1/(\gamma-1)}$$

and since $c_v = R/(\gamma - 1)$, the above can be written as

$$e^{\frac{\Delta s}{R}} = \left[\frac{1 + \frac{\gamma-1}{2} M_1^2 \sin^2\theta}{\frac{\gamma+1}{2} M_1^2 \sin^2\theta}\right]^{\gamma/(\gamma-1)} \left[\frac{2\gamma}{\gamma+1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1}\right]^{1/(\gamma-1)}$$

So

$$\frac{\Delta s}{R} = \frac{1}{(\gamma-1)} \ln\left[\left(\frac{1 + \frac{\gamma-1}{2} M_1^2 \sin^2\theta}{\frac{\gamma+1}{2} M_1^2 \sin^2\theta}\right)^{\gamma} \left(\frac{2\gamma}{\gamma+1} M_1^2 \sin^2\theta - \frac{\gamma-1}{\gamma+1}\right)\right]$$

Or

$$\frac{\Delta s}{c_v} = \ln\left[\left(\frac{2\gamma}{\gamma+1}M_1^2\sin^2\theta - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_1^2\sin^2\theta}\right)^\gamma\right]$$

Problem 14. – Repeat the computations of Example 6.2. However, instead of using the successive substitution method proposed by Collar, and described in Section 6.3, solve the problem using the Newton-Raphson method.

γ	M_1	A	B	C	B - AC	1^{rst} guess
1.3	2.0	3.0000	2.9143	0.8870	0.2534	1.7321

Newton-Raphson Method

iteration	x_{old}	f	fprime	x_{new}
1	1.7321	2.9143	9.0725	1.4108
2	1.4108	0.5945	5.4740	1.3022
3	1.3022	0.0591	4.3974	1.2888
4	1.2888	0.0009	4.2690	1.2886
5	1.2886	0.0000	4.2671	1.2886

Problem 15. – For the two-dimensional case shown in Figure P6.15, determine M_3 and p_3. $\gamma = 1.4$.

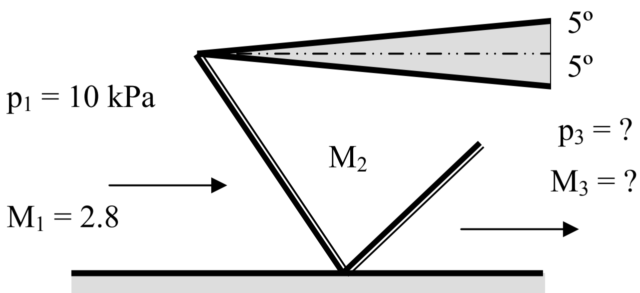


Figure P6.15

$$\left.\begin{array}{l} M_1 = 2.8 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta = 24.6427° \\ M_2 = 2.5677 \\ \dfrac{p_2}{p_1} = 1.4235 \end{array}$$

$$\left.\begin{array}{l} M_2 = 2.5677 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta = 26.7305° \\ M_3 = 2.3542 \\ \dfrac{p_3}{p_2} = 1.3895 \end{array}$$

$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_1}p_1 = (1.3895)(1.4235)10 = 19.7795\,\text{kPa}$$

Problem 16. – Prove that: (a) at the minimum shock angle, $M_2 = M_1$ and (b) at the maximum value of the shock angle, Eq.(6.17) becomes Eq.(4.9)

$$M_2^2 = \frac{1+\dfrac{\gamma-1}{2}M_1^2}{\gamma M_1^2\sin^2\theta - \dfrac{(\gamma-1)}{2}} + \frac{M_1^2\cos^2\theta}{1+\dfrac{\gamma-1}{2}M_1^2\sin^2\theta} \qquad \textbf{(6.17)}$$

(a) *at the minimum shock angle*

The minimum shock angle is the angle of a Mach wave for which $\theta = \sin^{-1}(1/M_1)$. Accordingly, $(M_1\sin\theta)^2 = 1$ and $(M_1\cos\theta)^2 = M_1^2 - 1$. Therefore,

$$M_2^2 = \frac{1+\dfrac{\gamma-1}{2}M_1^2}{\gamma M_1^2\sin^2\theta - \dfrac{(\gamma-1)}{2}} + \frac{M_1^2\cos^2\theta}{1+\dfrac{\gamma-1}{2}M_1^2\sin^2\theta} = \frac{1+\dfrac{\gamma-1}{2}M_1^2}{\gamma - \dfrac{(\gamma-1)}{2}} + \frac{M_1^2 - 1}{1+\dfrac{\gamma-1}{2}}$$

$$= \frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}M_1^2 + \frac{2}{\gamma+1}M_1^2 - \frac{2}{\gamma+1} = M_1^2$$

(b) *at the maximum value of the shock angle*

At the maximum shock angle $\theta = \pi/2$ so that $(M_1\sin\theta)^2 = M_1^2$ and $(M_1\cos\theta)^2 = 0$. Therefore,

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2\sin^2\theta - \frac{(\gamma-1)}{2}} + \frac{M_1^2\cos^2\theta}{1+\frac{\gamma-1}{2}M_1^2\sin^2\theta} = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{(\gamma-1)}{2}} + 0$$

$$= \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}$$

which is Eq.(4.9).

Problem 17. –Develop Prandtl's relation for oblique shocks from conservation principles.

Begin by writing the energy equation, Eq.(6.5c), as

$$h_o = \frac{a_o^2}{(\gamma-1)} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1} + \frac{V_{n1}^2 + V_t^2}{2} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2} + \frac{V_{n2}^2 + V_t^2}{2}$$

Thus,

$$p_2 = \rho_2\left[\frac{a_o^2}{\gamma} - \left(\frac{\gamma-1}{2\gamma}\right)\left(V_{n2}^2 + V_t^2\right)\right]$$

$$p_1 = \rho_1\left[\frac{a_o^2}{\gamma} - \left(\frac{\gamma-1}{2\gamma}\right)\left(V_{n1}^2 + V_t^2\right)\right]$$

From the momentum equation, Eq.(6.5b),

$$p_2 - p_1 = \rho_1 V_{n1}^2 - \rho_2 V_{n2}^2$$

Combining these yields

$$\rho_2\left[\frac{a_o^2}{\gamma} - \left(\frac{\gamma+1}{2\gamma}\right)V_{n2}^2 - \left(\frac{\gamma-1}{2\gamma}\right)V_t^2\right] = \rho_1\left[\frac{a_o^2}{\gamma} - \left(\frac{\gamma+1}{2\gamma}\right)V_{n1}^2 - \left(\frac{\gamma-1}{2\gamma}\right)V_t^2\right]$$

Rearrangement gives

$$\frac{\rho_1 V_{n1}^2 - \rho_2 V_{n2}^2}{\rho_2 - \rho_1} = \frac{2}{\gamma+1} a_o^2 - \left(\frac{\gamma-1}{\gamma+1}\right) V_t^2$$

Using the continuity equation, Eq.(6.5a), the expression can be simplified to obtain Prandtl's relation for an oblique shock wave

$$V_{n1} V_{n2} = \frac{2}{\gamma+1} a_o^2 - \left(\frac{\gamma-1}{\gamma+1}\right) V_t^2$$

Problem 18. – The largest deflection angle for the limiting upstream Mach number, $M_1 \to \infty$, can be found by differentiating Eq.(6.26), setting the result to zero and then solving for θ. In other words, verify that Eq.(6.27) is correct.

From Example 6.4 it was shown that

$$\tan\delta = \frac{\sin 2\theta}{\gamma + \cos 2\theta} \tag{6.26}$$

Therefore,

$$\frac{d(\tan\delta)}{d\theta} = \frac{2(\gamma + \cos 2\theta)\cos 2\theta - \sin 2\theta(-\sin 2\theta)2}{(\gamma + \cos 2\theta)^2} = 0$$

Cancel the 2 and rewrite the numerator as

$$\gamma\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta = \gamma\cos 2\theta + 1 = 0$$

Therefore,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = -\frac{1}{\gamma}$$

So that

$$\sin^2\theta = \frac{\gamma+1}{2\gamma}$$

Problem 19. – In general, the angle of incidence, θ_i, and the angle of reflection, θ_r, of an oblique shock reflected from a flat surface are not equal. However, see Refs. 8 and 9, there is an angle θ^* such that the two angles are equal. Also, if $\theta_i < \theta^*$, then $(\theta_r - \delta) < \theta_i$, and if $\theta_i > \theta^*$, then $(\theta_r - \delta) > \theta_i$. Computationally verify that for $M_1 = 2$, 3 and 4 at $\gamma = 1.4$, the angle of incidence and the angle of reflection of an oblique shock reflected from a flat surface will be equal if

$$\theta_i = \theta^* = \frac{1}{2}\cos^{-1}\left(\frac{\gamma - 1}{2}\right)$$

At $M_1 = 2$ and $\gamma = 1.4$, the computations yield the following values

γ	M_1	θ(deg)	δ(deg)	M_2
1.4	2.0	-39.2315	-9.9242	1.6433
	M_2	**θ(deg)**	**δ(deg)**	**M_3**
	1.6433	49.1557	9.9242	1.2910
	incidence	reflection	= (θ - δ)	
	-39.231520	39.231520		

At $M_1 = 3$ and $\gamma = 1.4$, the computations yield the following values

γ	M_1	θ(deg)	δ(deg)	M_2
1.4	3.0	-39.2315	-21.2229	1.9282
	M_2	**θ(deg)**	**δ(deg)**	**M_3**
	1.9282	60.4544	21.2229	1.0221
	incidence	reflection	= (θ - δ)	
	-39.231520	39.231520		

Note the flow in region 3 is just barely supersonic

At $M_1 = 4$ and $\gamma = 1.4$, the computations yield the following values

γ	M_1	θ(deg)	δ(deg)	M_2
1.4	4.0	-39.2315	-25.6060	2.1656
	M_2	**θ(deg)**	**δ(deg)**	**M_3**
	2.1656	64.3940	25.6060	0.9349
	incidence	reflection	= (θ - δ)	
	-39.231520	38.788033		

And as seen the incident and reflected angles are not equal. Also M_3 is subsonic, which is possible for a weak shock. However, when we use the shock shock solution instead, the following is obtained

γ	M_1	θ(deg)	δ(deg)	M_2
1.4	4.0	-39.2315	-25.6060	2.1656
	M_2	**θ(deg)**	**δ(deg)**	**M_3**
	2.1656	64.8375	25.6060	0.9239
	incidence	reflection	= (θ - δ)	
	-39.231520	39.231534		

Problem 20. – Complete the computations of Example 6.7, i.e., use the computed flow angles to determine the deflection angles, and with M_1 and M_2, determine all parameters in regions 3 and 4 of Figure 6.18.

region 1 to region 3							
γ	$\mathbf{M_1}$	α_1	$\mathbf{p_2/p_1}$	ρ_3/ρ_1	$\mathbf{T_3/T_1}$	$\mathbf{p_{o3}/p_{o1}}$	
1.4	2.0	0.00	1	1	1	1	
	δ_{13}	θ	$\mathbf{p_3/p_1}$	ρ_3/ρ_1	$\mathbf{T_3/T_1}$	$\mathbf{p_{o3}/p_{o1}}$	$\mathbf{M_3}$
	-5.7977	-35.0485	1.3723	1.2525	1.0957	0.9968	1.7928
region 2 to region 4							
γ	$\mathbf{M_2}$	α_2	$\mathbf{p_2/p_1}$	ρ_2/ρ_1	$\mathbf{T_2/T_1}$	$\mathbf{p_{o2}/p_{o1}}$	
1.4	3.0	-10.00	1	1	1	1	
	δ_{24}	θ	$\mathbf{p_4/p_2}$	ρ_4/ρ_2	$\mathbf{T_4/T_2}$	$\mathbf{p_{o4}/p_{o2}}$	$\mathbf{M_4}$
	4.2023	22.5101	1.3723	1.2525	1.0957	0.9968	2.7889

δ_{13}	-5.7977	δ_{24}	4.2023
α_1	0.00000	α_2	-10.0000
α_3	-5.7977	α_4	-5.7977
$\delta_{13}=\alpha_3-\alpha_1$		$\delta_{24}=\alpha_4-\alpha_2$	

Problem 21. – Derive the pressure-deflection equation, i.e., Eq.(6.30).

The expression for the pressure ratio across an oblique shock, is given in Eq.(6.10)

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\theta}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$$

This can be rearranged to obtain

$$M_1^2 \sin^2\theta - 1 = \left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1} - 1\right)$$

The following identity is also used in this development

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \pm\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}} = \pm\sqrt{\frac{M_1^2\left(1-\sin^2\theta\right)-1+1}{M_1^2\sin^2\theta-1+1}}$$

$$= \pm\sqrt{\frac{\left(M_1^2-1\right)-\left(M_1^2\sin^2\theta-1\right)}{\left(M_1^2\sin^2\theta-1\right)+1}}$$

Now the deflection angle is connected to the shock wave angle and the upstream Mach number by Eq.(6.18)

$$\tan\delta = \cot\theta\frac{M_1^2\sin^2\theta-1}{\frac{\gamma+1}{2}M_1^2-\left(M_1^2\sin^2\theta-1\right)} \qquad \textbf{(6.18)}$$

Dividing the expression on the right into two pieces

$$\frac{M_1^2\sin^2\theta-1}{\frac{\gamma+1}{2}M_1^2-\left(M_1^2\sin^2\theta-1\right)} = \frac{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1}-1\right)}{\frac{\gamma+1}{2}M_1^2-\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1}-1\right)} = \frac{\left(\frac{p_2}{p_1}-1\right)}{1+\gamma M_1^2-\frac{p_2}{p_1}}$$

$$\cot\theta = \pm\sqrt{\frac{\left(M_1^2-1\right)-\left(M_1^2\sin^2\theta-1\right)}{\left(M_1^2\sin^2\theta-1\right)+1}} = \pm\sqrt{\frac{\left(M_1^2-1\right)-\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1}-1\right)}{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_2}{p_1}-1\right)+1}}$$

$$= \pm\sqrt{\frac{\left(\frac{2\gamma}{\gamma+1}\right)\left(M_1^2-1\right)-\frac{p_2}{p_1}+1}{\frac{p_2}{p_1}-1+\left(\frac{2\gamma}{\gamma+1}\right)}} = \pm\sqrt{\frac{\left(\frac{2\gamma}{\gamma+1}\right)M_1^2-\frac{\gamma-1}{\gamma+1}-\frac{p_2}{p_1}}{\frac{\gamma-1}{\gamma+1}+\frac{p_2}{p_1}}}$$

Combining these two pieces yields the *pressure-deflection* equation

$$\tan\delta = \tan(\alpha_2 - \alpha_1) = \pm \frac{\dfrac{p_2}{p_1} - 1}{\left(1 + \gamma M_1^2\right) - \dfrac{p_2}{p_1}} \sqrt{\frac{\left(\dfrac{2\gamma}{\gamma+1}M_1^2 - \dfrac{\gamma-1}{\gamma+1}\right) - \dfrac{p_2}{p_1}}{\left(\dfrac{\gamma-1}{\gamma+1}\right) + \dfrac{p_2}{p_1}}} \qquad \textbf{(6.30)}$$

Problem 22. – Repeat the computations of Example 6.8 to find the angle the slip line makes with the horizontal for $\gamma = 1.4$ and 1.667. How does the angle vary with γ?

Results of the computations for all three specific heat ratios (1.3, 1.4 and 5/3) is as follows:

$\gamma = 1.3$

region 1 to region 2 to region 4								
γ	M_1	δ_{1-2}	θ_{1-2}	p_2/p_1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}	
1.3000	3.5000	10.0000	23.9901	2.1587	1.7862	1.2085	0.9500	
	M_2	δ_{2-4}	θ_{2-4}	p_4/p_2	ρ_4/ρ_2	T_4/T_2	p_{o4}/p_{o2}	M_4
	2.9976	-14.8926	-31.4440	2.6339	2.0575	1.2802	0.9032	2.3580
region 1 to region 3 to region 5								
γ	M_1	δ_{1-3}	θ_{1-3}	p_3/p_1	ρ_3/ρ_1	T_3/T_1	p_{o3}/p_{o1}	
1.3000	3.5000	-15.0000	-28.5011	3.0227	2.2615	1.3366	0.8598	
	M_3	δ_{3-5}	θ_{3-5}	p_5/p_3	ρ_5/ρ_3	T_5/T_3	p_{o5}/p_{o3}	M_5
	2.7361	10.1074	29.1782	1.8810	1.6152	1.1646	0.9719	2.3422

Downstream flow angles			
δ_{24}	-14.8926	δ_{35}	10.1074
α_2	10.00000	α_3	-15.0000
α_4	-4.8926	α_5	-4.8926
$\delta_{24}=\alpha_4-\alpha_2$		$\delta_{35}=\alpha_5-\alpha_3$	

$\gamma = 1.4$

region 1 to region 2 to region 4								
γ	M_1	δ_{1-2}	θ_{1-2}	p_2/p_1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}	
1.4000	3.5000	10.0000	24.3840	2.2693	1.7675	1.2839	0.9463	
	M_2	δ_{2-4}	θ_{2-4}	p_4/p_2	ρ_4/ρ_2	T_4/T_2	p_{o4}/p_{o2}	M_4
	2.9044	-14.8780	-32.8511	2.7293	1.9905	1.3711	0.9042	2.1906
region 1 to region 3 to region 5								
γ	M_1	δ_{1-3}	θ_{1-3}	p_3/p_1	ρ_3/ρ_1	T_3/T_1	p_{o3}/p_{o1}	
1.4000	3.5000	-15.0000	-29.1916	3.2331	2.2093	1.4634	0.8528	
	M_3	δ_{3-5}	θ_{3-5}	p_5/p_3	ρ_5/ρ_3	T_5/T_3	p_{o5}/p_{o3}	M_5
	2.6053	10.1220	30.8502	1.9157	1.5784	1.2137	0.9726	2.1708

Downstream flow angles			
δ_{24}	-14.8780	δ_{35}	10.1220
α_2	10.00000	α_3	-15.0000
α_4	-4.8780	α_5	-4.8780
$\delta_{24}=\alpha_4-\alpha_2$		$\delta_{35}=\alpha_5-\alpha_3$	

$\gamma = 5/3$

region 1 to region 2 to region 4								
γ	M_1	δ_{1-2}	θ_{1-2}	p_2/p_1	ρ_2/ρ_1	T_2/T_1	p_{o2}/p_{o1}	
1.6667	3.5000	10.0000	25.4710	2.5820	1.7211	1.5002	0.9366	
	M_2	δ_{2-4}	θ_{2-4}	p_4/p_2	ρ_4/ρ_2	T_4/T_2	p_{o4}/p_{o2}	M_4
	2.6768	-14.8530	-36.9465	2.9858	1.8528	1.6115	0.9057	1.8187
region 1 to region 3 to region 5								
γ	M_1	δ_{1-3}	θ_{1-3}	p_3/p_1	ρ_3/ρ_1	T_3/T_1	p_{o3}/p_{o1}	
1.6667	3.5000	-15.0000	-31.1387	3.8446	2.0879	1.8414	0.8356	
	M_3	δ_{3-5}	θ_{3-5}	p_5/p_3	ρ_5/ρ_3	T_5/T_3	p_{o5}/p_{o3}	M_5
	2.2982	10.1470	35.7646	2.0052	1.5022	1.3349	0.9740	1.7900

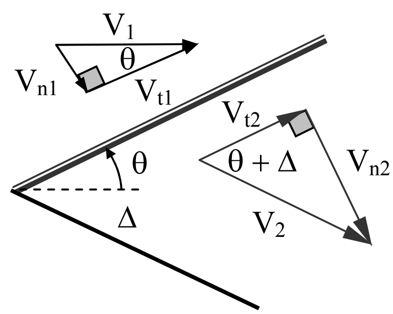
Downstream flow angles			
δ_{24}	-14.8530	δ_{35}	10.1470
α_2	10.00000	α_3	-15.0000
α_4	-4.8530	α_5	-4.8530
$\delta_{24}=\alpha_4-\alpha_2$		$\delta_{35}=\alpha_5-\alpha_3$	

As can be seen, the angle of the slip line ($\alpha_4 = \alpha_5$) is diminished slightly as γ is increased.

Chapter Seven

PRANDTL-MEYER FLOW

Problem 1. – Use a trigonometric development to demonstrate that for an expansion flow around a convex corner, $V_{n2} > V_{n1}$ (see Figure 7.2 in Section 7.2).



Now

$$\tan\theta = \frac{V_{n1}}{V_{t1}}$$

$$\tan(\theta + \Delta) = \frac{V_{n2}}{V_{t2}}$$

The momentum equation in the tangential direction reveals that $V_{t1} = V_{t2}$. Therefore, equating the above brings

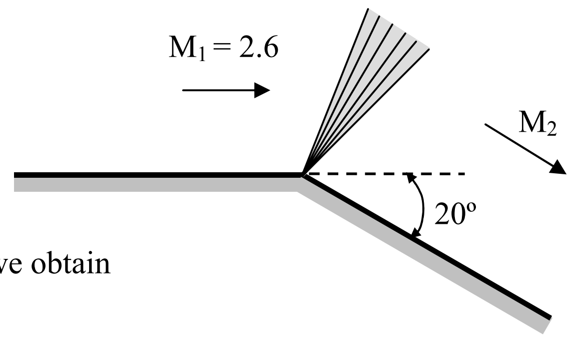
$$V_{n2} = \frac{\tan(\theta + \Delta)}{\tan\theta} V_{n1}$$

Since, $\tan(\theta + \Delta) > \tan\theta$, it follows that $V_{n2} > V_{n1}$.

Problem 2. – A uniform supersonic flow of air ($\gamma = 1.4$) at Mach 2.6, with stagnation pressure of 5 MPa and stagnation temperature of 1000 K, expands around a 20° convex corner. Determine the downstream Mach number, the stagnation pressure and temperature, and the static pressure and temperature.

$M_1 = 2.6, \quad \nu_1 = 41.4147°$

$\nu_2 = \nu_1 + 20 = 61.4147°$



Using the solver developed in Example 7.1, we obtain

$M_2 = 3.6878$

$p_{02} = p_{o1} = 5$ MPa

$T_{o2} = T_{o1} = 1000$ K

Now from the isentropic flow relations

$$\frac{T_2}{T_{o2}} = 0.2688, \quad \text{so} \quad T_2 = 268.8 \text{ K}$$

$$\frac{p_2}{p_{o2}} = 0.0101, \quad p_2 = 5(0.0101) = 0.0505 \text{ MPa} = 50.5 \text{ kPa}$$

Problem 3. – Integrate Eq.(7.7). To accomplish this first use a transformation in which $x^2 = M^2 - 1$ and then use the method of partial fractions to break the transformed integrand into two groups of terms, which may be integrated using:

$$\int \frac{du}{a^2 + b^2 u^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bu}{a}\right)$$

Now

$$d\alpha = -\left(\frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma - 1}{2}M^2}\right)\frac{dM}{M}$$

Let $x^2 = M^2 - 1$ or $M^2 = 1 + x^2$. Therefore, $2MdM = 2xdx$ or

$$\frac{dM}{M} = \frac{xdx}{1 + x^2}$$

Perform the transformation of the terms on the right to obtain

$$-\,d\alpha = \frac{x}{1+\frac{\gamma-1}{2}\left(1+x^2\right)}\frac{xdx}{1+x^2} = \left[\frac{2x^2}{\left[(\gamma+1)+(\gamma-1)x^2\right]}\right]\frac{1}{\left(1+x^2\right)}dx$$

Next use partial fractions to divide the right hand side into two groups of terms

$$\frac{A}{[(\gamma+1)+(\gamma-1)x^2]}+\frac{B}{(1+x^2)}=\frac{2x^2}{[(\gamma+1)+(\gamma-1)x^2](1+x^2)}$$

So,

$$A + Ax^2 + (\gamma+1)B + (\gamma-1)Bx^2 = 2x^2$$

$$A + (\gamma-1)B = 2$$
$$A + (\gamma+1)B = 0$$

Solving this pair yields: $A = \gamma + 1$ and $B = -1$. Thus, the transformed equation can be arranged into two groups and leads to the following two integrals:

$$-\int\left(\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\right)\frac{dM}{M} = -\int\frac{\gamma+1}{[(\gamma+1)+(\gamma-1)x^2]}dx - \int\frac{1}{(1+x^2)}dx$$

$$= -\int\frac{1}{[1+\frac{(\gamma-1)}{(\gamma+1)}x^2]}dx - \int\frac{1}{(1+x^2)}dx$$

Making use of the given integral identity we get

$$-\int\left(\frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\right)\frac{dM}{M} = -\left\{\sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right] - \tan^{-1}\left(\sqrt{M^2-1}\right)\right\}$$

Problem 4. – A reservoir containing air ($\gamma = 1.4$) at 2 MPa is connected to ambient air at 101 kPa through a converging-diverging nozzle designed to produce flow at Mach 2.0, with axial flow at the nozzle exit plane (Figure P7.4). Under these conditions, the nozzle is underexpanded, with a Prandtl Meyer expansion fan at the exit. Find the flow direction after the initial expansion fan. How does this turning angle affect the net axial thrust forces exerted by the fluid on the nozzle?

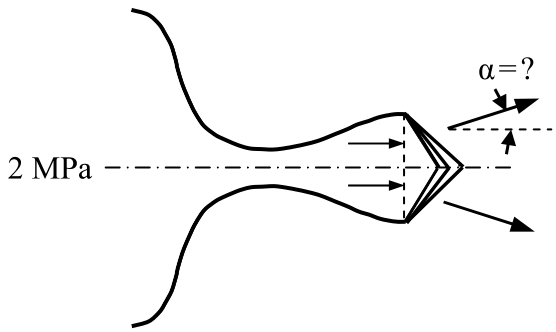


Figure P7.4

At $M_1 = 2.0$, $\dfrac{p_1}{p_{o1}} = 0.1278$, $\nu_1 = 26.3798°$

$$\frac{p_2}{p_{o1}} = \frac{101}{2000} = 0.0505, \quad \text{so} \quad M_2 = 2.595, \quad \text{and therefore } \nu_2 = 41.3044°$$

The angle through which the flow turned is:

$$\alpha = \nu_2 - \nu_1 = 41.3044 - 26.3798 = 14.9246°$$

The turning does not affect thrust, because the expansion occurs outside nozzle.

Problem 5. – Develop a computer program that will yield values of ν and μ versus M for Prandtl-Meyer flow for $\gamma = 1.3$ over the range M = 1.0 to M = 2.5, using Mach number increments of 0.1.

A table of the Prandtl-Meyer function and wave angle versus Mach number for γ = 1.3

M	ν (rad)	ν (deg)	μ (rad)	μ (deg)
1.000	0.0000	0.0000	1.5708	90.0000
1.100	0.0244	1.4004	1.1411	65.3800
1.200	0.0654	3.7454	0.9851	56.4427
1.300	0.1138	6.5230	0.8776	50.2849
1.400	0.1665	9.5414	0.7956	45.5847
1.500	0.2215	12.6928	0.7297	41.8103
1.600	0.2777	15.9089	0.6751	38.6822
1.700	0.3341	19.1436	0.6289	36.0319
1.800	0.3903	22.3645	0.5890	33.7490
1.900	0.4459	25.5491	0.5543	31.7569
2.000	0.5006	28.6809	0.5236	30.0000
2.100	0.5541	31.7483	0.4963	28.4369
2.200	0.6064	34.7433	0.4719	27.0357
2.300	0.6573	37.6605	0.4498	25.7715
2.400	0.7068	40.4962	0.4298	24.6243
2.500	0.7548	43.2486	0.4115	23.5782

Problem 6. – A uniform supersonic flow of a perfect gas with $\gamma = 1.3$ and Mach number 3.0 expands around a 5° convex corner. Determine the downstream Mach number, ratio of downstream to upstream velocity, and ratio of downstream to upstream stagnation temperature.

$$\text{For } M_1 = 3, \ \ \gamma = 1.3, \ \ \nu_1 = 55.7584°\text{, and } \frac{T_1}{T_{o1}} = 0.4255$$

$$\nu_2 = \nu_1 + 5 = 60.7584°, \quad \text{so} \quad M_2 = 3.2261\text{, and } \frac{T_2}{T_{o2}} = 0.3904$$

$$\frac{T_{o2}}{T_{o1}} = 1.0, \ \ \frac{v_2}{v_1} = \left(\frac{M_2}{M_1}\right)\left(\frac{a_2}{a_1}\right) = \left(\frac{M_2}{M_1}\right)\sqrt{\frac{T_2}{T_1}} = \frac{3.2261}{3.00}\sqrt{\frac{0.3904}{0.4255}} = 1.0301$$

Problem 7. – For flow at Mach 2.5 and $\gamma = 1.4$ over the symmetrical protrusion shown in Figure P7.5, find M_2, M_3, M_4, T_2, T_3, and T_4.

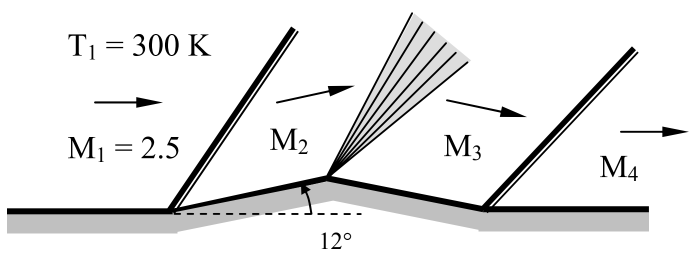


Figure P7.7

$$\left.\begin{array}{l} M_1 = 2.5 \\ \delta = 12° \\ \dfrac{T_1}{T_{o1}} = 0.4444 \end{array}\right\} \begin{array}{l} \theta = 33.8016° \\ M_2 = 2.0022 \\ \nu_2 = 26.4404° \\ \dfrac{T_2}{T_{o2}} = 0.5550 \end{array}$$

$$\nu_3 = \nu_2 + \delta_{1-2} = 26.4404 + 24 = 50.4404°$$

$$M_3 = 3.0356$$

$$\left.\begin{array}{l} M_3 = 3.0356 \\ \delta = 12° \\ \dfrac{T_3}{T_{o3}} = 0.3517 \end{array}\right\} \begin{array}{l} \theta = 28.9964° \\ M_4 = 2.4338 \\ \dfrac{T_4}{T_{o4}} = 0.4577 \end{array}$$

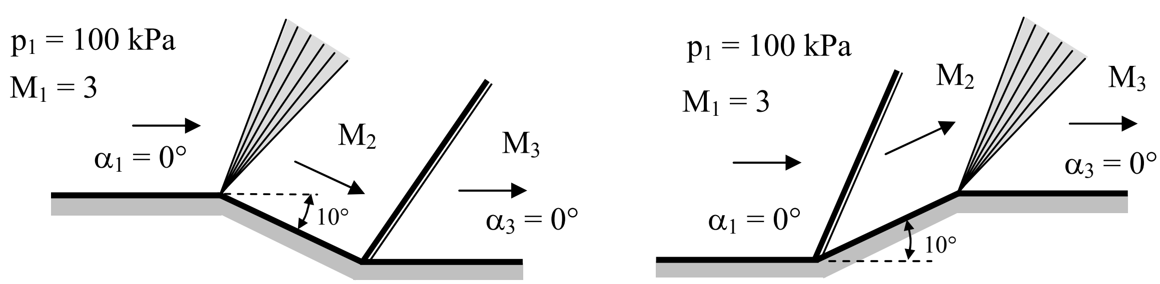
$$T_{o1} = T_{o2} = T_{o3} = T_{o4} = \left(\frac{T_{o1}}{T_1}\right)T_1 = \left(\frac{1}{0.4444}\right)300 = 675.0675\,\text{K}$$

$$T_2 = 0.5550(675.0675) = 374.6625\,\text{K}$$

$$T_3 = 0.3517(675.0675) = 237.4212\,\text{K}$$

$$T_4 = 0.4577(675.0675) = 308.9784\,\text{K}$$

Problem 8. – A uniform supersonic flow of a perfect gas with $\gamma = 1.4$, Mach number 3.0 and an upstream static pressure of 100kPa flows over a geometry as shown in P7.8. Determine the downstream static pressure for both profiles.



(a) Expansion Fan-Oblique Shock Geometry (b) Oblique Shock- Expansion Fan Geometry

Figure P7.8

Divide the flow field of both cases shown in Figure P7.8 into 3 regions of uniform flow with region 1 on the left and region 3 on the right.

Case (a)

From the isentropic and Prandtl-Meyer relations at $\gamma = 1.4$ and $M_1 = 3.0$

$$\frac{p_1}{p_{o1}} = 0.02722, \qquad \nu_1 = 49.7573°$$

Region 2 is reached by passing through an expansion fan in which the flow is turned 10º. Therefore,

$$\nu_2 = \nu_1 + 10 = 59.7573°, \quad \text{so} \quad M_2 = 3.5783 \text{ and } \frac{p_2}{p_{o2}} = 0.01174$$

Region 3 is reached by passing through an oblique shock in which the flow is turned back 10º. Therefore, using the oblique shock relations

$$M_3 = 2.9653, \; \frac{p_3}{p_2} = 2.3049 \text{ and } \frac{p_{o3}}{p_{o1}} = \frac{p_{o3}}{p_{o2}}\frac{p_{o2}}{p_{o1}} = (0.9433)(1) = 0.9433$$

$$p_3 = \frac{p_3}{p_2}\frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o1}}\frac{p_{o1}}{p_1}p_1 = (2.3049)(0.01174)(1)\left(\frac{1}{0.02722}\right)100 = 99.4105\,\text{kPa}$$

Case (b)

Region 2 is reached by passing through an oblique shock in which the flow is turned through 10º. Therefore, using the oblique shock relations with $\gamma = 1.4$, $M_1 = 3.0$ and $\delta = 10º$,

$$M_2 = 2.5050, \; \frac{p_2}{p_1} = 2.0545 \text{ and } \frac{p_{o2}}{p_{o1}} = 0.9631$$

From the isentropic and Prandtl-Meyer relations at $\gamma = 1.4$ and $M_2 = 2.5050$

$$\frac{p_2}{p_{o2}} = 0.05807, \qquad \nu_2 = 39.2402°$$

Region 3 is reached by passing through an expansion fan in which the flow is turned 10º. Therefore,

$$\nu_3 = \nu_2 + 10 = 49.2402°, \quad \text{so} \quad M_3 = 2.9733 \text{ and } \frac{p_3}{p_{o3}} = 0.02834$$

$$p_3 = \frac{p_3}{p_{o3}}\frac{p_{o3}}{p_{o2}}\frac{p_{o2}}{p_2}\frac{p_2}{p_1}p_1 = (0.02834)(1)\left(\frac{1}{0.05807}\right)(2.50545)100 = 100.2661\,\text{kPa}$$

Problem 9. – A two-dimensional, flat plate is inclined at a positive angle of attack in a supersonic air stream of Mach 2.0 (Figure P7.6). Below the plate, an oblique shock wave

starts at the leading edge, making an angle of 42° with the stream direction. On the upper side, an expansion occurs at the leading edge.
(a) Find the angle of attack, AoA, of the plate.
(b) What is the pressure on the lower surface of the plate?
(c) What is the pressure on the upper surface of the plate?

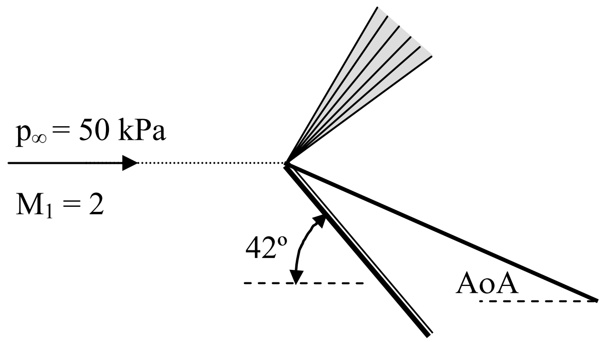


Figure P7.9

From the oblique shock relations,

(a) $\left.\begin{array}{l}\theta = 42° \text{ or } 180 - 42 = 138° \\ M_1 = 2.0\end{array}\right\} \delta_{1\text{-}2} = -12.3589°$

(b) $M_n = M_1 \sin\theta = 2\sin 42° = 1.3383$

$$\frac{p_2}{p_1} = \frac{p_2}{p_\infty} = \frac{2\gamma M_1^2 \sin^2\theta}{\gamma+1} - \frac{\gamma-1}{\gamma+1} = \frac{2\gamma M_n^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} = 1.9228$$

$$p_2 = (1.9228)50 = 96.1383 \text{kPa}$$

(c) At M_1 we find, $\nu_1 = 26.3798°$ and since the flow on the top of the plate must be turned through the same amount as on the bottom, we may write

$$\nu_3 = \nu_1 + \delta_{1-2} = 26.3798 + 12.3589 = 38.7387°$$

With this value of the Prandtl-Meyer function, we find

$$M_3 = 2.4836$$

Now because the flow through the expansion fan is isentropic, i.e., $p_{o1} = p_{o3}$,

$$\frac{p_1}{p_{o1}} = \frac{p_\infty}{p_{o1}} = 0.1278 \text{ and } \frac{p_3}{p_{o3}} = 0.06004$$

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_{o3}}\frac{p_{o1}}{p_\infty} = \frac{0.06004}{0.1278} = 0.4698$$

$$p_3 = (0.4698)50 = 23.4900\,\text{kPa}$$

Problem 10. – A two-dimensional supersonic wing has the profile shown in Figure P7 .7. At zero angle of attack, determine the drag force on the wing per unit length of span at Mach 2 and at Mach 4. Repeat for the lift force. Take the maximum thickness of the airfoil to be 0.2m.

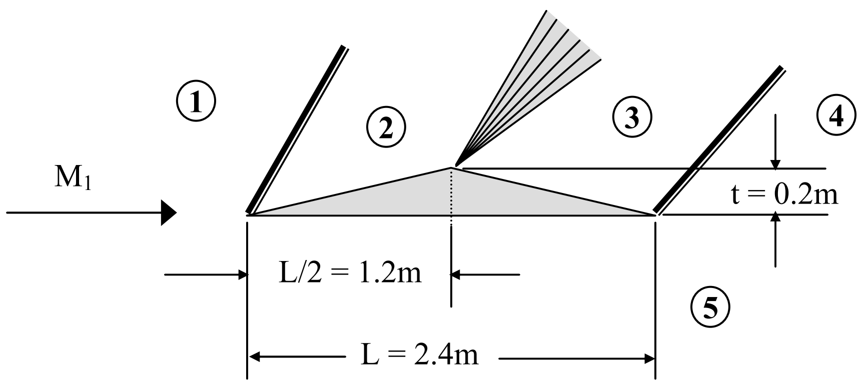


Figure P7.7

$M_1 = 2.0$ computations

$$M_1 = 2.0, \quad \delta = \tan^{-1}\left(\frac{t}{L/2}\right) = \tan^{-1}\left(\frac{0.2}{1.2}\right) = 9.46°$$

At this Mach number and deflection angle, the shock wave angle is found to be $\theta = 38.8°$ and $M_2 = 1.6604$, $\frac{p_2}{p_1} = 1.6604$. Furthermore, from the isentropic flow relations at M_2, we have $\frac{p_2}{p_{o2}} = 0.2150$ and the Prandtl-Meyer function at this Mach number is 16.6446º.
Now since the flow must be turned through 2δ in passing from region 2 to 3, we may write

$$\nu_3 = \nu_2 + 2\delta = 16.6446 + 2(9.4623) = 35.5692°$$

From this value we can determine the Mach number in region 3 to be $M_3 = 2.3518$. At this value we can return to the isentropic relations to find the static to total pressure ratio in region 3 to be $p_3/p_{o3} = 0.0737$. We are now in a position to compute the pressure on the rear side of the airfoil, i.e., p_3.

$$\frac{p_3}{p_2} = \left(\frac{p_3}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_2}\right) = (0.0737)(1)\left(\frac{1}{0.2150}\right) = 0.3428$$

$$p_1 = p_\infty = 20 \text{ kPa},$$
$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 = (1.6604)(20) = 33.208\,\text{kPa},$$
$$p_3 = \left(\frac{p_3}{p_2}\right)p_2 = (0.3428)(33.208) = 11.3837 \text{ kPa}$$

$$\text{Drag} = (p_2)\left(\frac{t}{\sin\delta}\right)\sin\delta - (p_3)\left(\frac{t}{\sin\delta}\right)\sin\delta = (p_2 - p_3)t = (33.2080 - 11.3837)0.2$$
$$= 4.3649\text{kN/m}$$

$$\text{Lift} = (p_1)(L) - (p_2)\left(\frac{t}{\sin\delta}\right)\cos\delta - (p_3)\left(\frac{t}{\sin\delta}\right)\cos\delta = p_1 L - (p_2 + p_3)\frac{t}{\cot\delta}$$
$$= p_1 L - (p_2 + p_3)\frac{L}{2} =$$
$$= (20)(2.4) - (33.2080 + 11.3837)(1.2) = -5.510\text{kN/m}$$

$M_1 = 4.0$ computations

At this Mach number and the deflection angle of 9.4623º, $\theta = 21.7505°$ and $M_2 = 3.3241$, $\frac{p_2}{p_1} = 2.3966$. Furthermore, at M_2, $\frac{p_2}{p_{o2}} = 0.016876$ and ν_2 = 55.6341º. Therefore,

$$\nu_3 = \nu_2 + 2\delta = 55.6341 + 2(9.4623) = 74.5587°$$

Hence, M_3 = 4.7575. The static to total pressure ratio in region 3 is p_3/p_{o3} = 0.00252. The pressure on the rear side of the airfoil, i.e., p_3 is computed as

$$\frac{p_3}{p_2} = \left(\frac{p_3}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_2}\right) = (0.00252)(1)\left(\frac{1}{0.016876}\right) = 0.14932$$

$$p_1 = p_\infty = 20 \text{ kPa},$$

$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 = (2.3966)(20) = 47.932\,\text{kPa},$$

$$p_3 = \left(\frac{p_3}{p_2}\right)p_2 = (0.14932)(33.208) = 7.1572 \text{ kPa}$$

$$\text{Drag} = (p_2)\left(\frac{t}{\sin\delta}\right)\sin\delta - (p_3)\left(\frac{t}{\sin\delta}\right)\sin\delta = (p_2 - p_3)t = (47.932 - 7.1572)0.2$$
$$= 8.155\text{kN/m}$$

$$\text{Lift} = (p_1)(L) - (p_2)\left(\frac{t}{\sin\delta}\right)\cos\delta - (p_3)\left(\frac{t}{\sin\delta}\right)\cos\delta = p_1 L - (p_2 + p_3)\frac{t}{\cot\delta}$$
$$= p_1 L - (p_2 + p_3)\frac{L}{2} =$$
$$= (20)(2.4) - (47.932 + 7.1572)(1.2) = -18.107\text{kN/m}$$

Problem 11. In Problem 10, a compression occurs at the trailing edge, with the resultant flows in regions (a) and (b) parallel (Figure P7.11). Is there any difference in pressure, velocity, or entropy between regions (a) and (b)? Discuss.

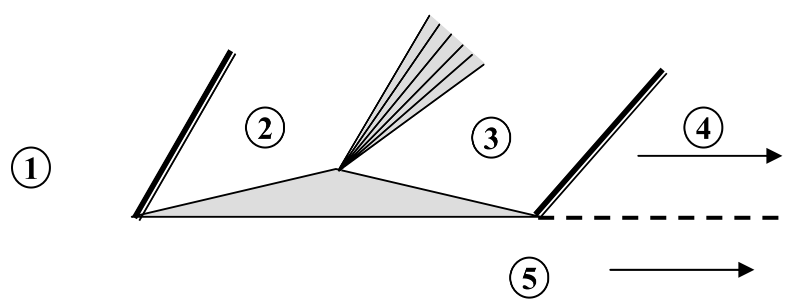


Figure P7.11

The flow over the top of the wing has gone through two shocks, and an isentropic expansion fan, whereas, the flow over the bottom has undergone no shocks. Therefore,

entropy (4) > entropy (5)

Consequently, a contact discontinuity or slip line separates the two regions. The flow direction in the two regions is the same and there can be no pressure difference between (4) and (5). However, there is a velocity difference between (4) and (5).

Problem 12. – A reservoir containing air at 10 MPa is discharged through a converging-diverging nozzle of area ratio 3.0. An expansion fan is observed at the exit, with the flow immediately downstream of the fan turned through an angle of 10°. Determine the pressure of the region into which the nozzle is exhausting, if the air can be assumed to behave as a perfect gas with constant $\gamma = 1.4$.

For the given area ratio: $\frac{A}{A^*} = 3.0$ we can determine the corresponding Mach number for the supersonic case to be $M_e = 2.6374$. At this Mach number, the Prandtl-Meyer function is found to be $\nu_e = 42.2498°$. After the exiting flow is turned through 10º the Prandtl-Meyer function is

$$\nu_b = \nu_e + 10 = 52.2498°$$

From this value, we can find the corresponding Mach number

$$M_b = 3.1325$$

$$\frac{p_b}{p_r} = \left(\frac{p_b}{p_{ob}}\right)\left(\frac{p_{ob}}{p_{oe}}\right)\left(\frac{p_{oe}}{p_r}\right) = (0.02234549)(1)(1) = 0.02234549$$

$$p_b = 0.02234549(10 \text{ MPa}) = 223.4549 \text{ kPa}$$

Problem 13. –Determine the value of γ for which $\nu_{max} = 180°$.

From Eq.(7.15)

$$\nu_{max} = 180° = \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)90°$$

So

$$2 = \sqrt{\frac{\gamma+1}{\gamma-1}} - 1$$

or

$$9 = \frac{\gamma+1}{\gamma-1}$$

Solving we find that: $8\gamma = 10$ or $\gamma = 1.25$.

Problem 14. – For the geometry shown in P7.15 along with the given values of the fan angle and the deflection angle, determine M_1 and M_2.

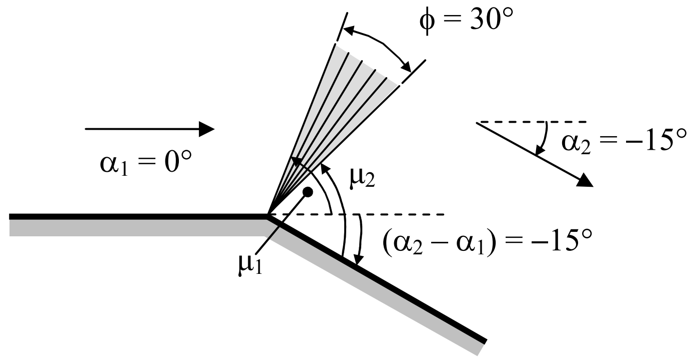


Figure P7.14

The solution of this problem requires a trial and error approach involving the following two equations

$$\phi = \mu_1 - \mu_2 + \Delta$$

$$\nu_2 = \nu_1 + \Delta$$

Since $\phi = 30^o$ and $\Delta = 15^o$, this pair of equations can be written as

$$15 = \nu_2 - \nu_1$$

$$15 = \mu_1 - \mu_2$$

Now since both μ_1 and ν_1 depend only on M_1 and since both μ_2 and ν_2 depend only on M_2, then the above pair represents two equations with two unknowns. One procedure to solve the pair is

1. assume an M_1,
2. determine ν_1 from the Prandlt-Meyer relation,
3. with ν_1 use the first expression to compute ν_2,
4. from ν_2 obtain M_2,
5. with M_2 we can determine μ_2,
6. with μ_2 use the second expression above to compute μ_1,
7. from μ_1 determine M_1, and
8. repeat the process until the computed M_1 value in step7 agrees with the assumed value in step 1.

The following table contains some of the computations from this process

M_1	ν_1	ν_2	M_2	μ_2	μ_1	M_1
2.0	26.38	41.38	2.5984	22.634	37.634	1.6377
1.6377						1.4822
1.4822						1.4142
1.4142						1.3848
1.3848						1.3722
1.3722						1.3669
1.3669						1.3646
1.3646						1.3637
1.3637						1.3633
1.3633						1.3631
1.3631						1.3630
1.3630	7.9286	22.929	1.8768	32.196	47.196	1.3630

Therefore, $M_1 = 1.3630$ and $M_2 = 1.8768$.

Problem 15. – For the geometry of Figure P7.14, and for given values of the wall turning angle, Δ, and the static pressure ratio across the expansion fan, p_2/p_1, define a process that will yield M_1 and M_2. Use the process to solve for these Mach numbers if $p_2 = 0.4p_1$ and $\Delta = 10°$. Take $\gamma = 1.4$.

The following outlines a computational process

1. assume an M_1,
2. determine p_1/p_o from the isentropic pressure relation,
3. compute $p_2/p_o = (p_2/p_1)(p_1/p_o)$,
4. obtain M_2 from p_2/p_o,
5. determine ν_2 from M_2,
6. compute $\nu_1 = \nu_2 - \Delta$,
7. determine M_1 from ν_1, and
8. repeat the process until the computed M_1 value in step7 agrees with the assumed value in step 1.

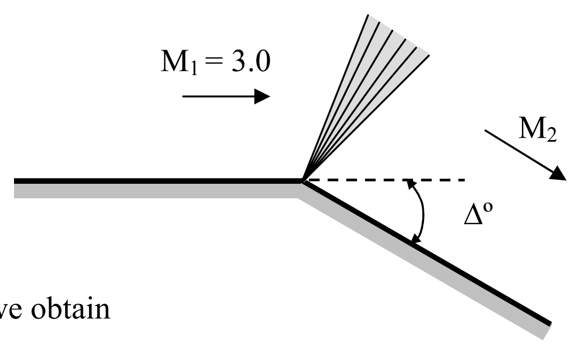
The results of the computations are contained in the following table

M_1	p_1/p_o	p_2/p_o	M_2	ν_2	ν_1	M_1
2.0000	0.1278	0.051122	2.5872	41.1251	31.1251	2.1767
2.5000	0.0585	0.023411	3.1011	51.6699	41.6699	2.6114
3.0000	0.0272	0.010889	3.6318	60.5757	50.5757	3.0428
3.2000	0.0202	0.0081	3.8472	63.7088	53.7088	3.2134
3.3000	0.0175	0.0070	3.9554	65.1907	55.1907	3.2982
3.2982	0.0175	0.007009	3.9535	65.1649	55.1649	3.2967
3.2967	0.0176	0.007025	3.9518	65.1421	55.1421	3.2954
3.2954	0.0176	0.007038	3.9504	65.1235	55.1235	3.2943
3.2943	0.0176	0.007049	3.9492	65.1079	55.1079	3.2934
3.2934	0.0176	0.007058	3.9483	65.0951	55.0951	3.2926
3.2926	0.0177	0.007066	3.9474	65.0837	55.0837	3.2920
3.2900	0.0177	0.007093	3.9446	65.0455	55.0455	3.2897
3.2895	0.0177	0.007098	3.9441	65.0384	55.0384	3.2893
3.2891	0.0178	0.007102	3.9436	65.0327	55.0327	3.2890

The method produces $M_1 = 3.289$ and $M_2 = 3.9436$, however, it converges very slowly.

Problem 16. – A gas ($\gamma = 1.44$, $R = 256$ J/kg·K) flows towards a convex corner with $M_1 = 3$ and $T_1 = 300$ K. Determine the downstream Mach number M_2 and the downstream velocity V_2 if the wall is turned 15°. Repeat the calculations if the wall is turned 30°.

Case (a) $\Delta = 15°$



$$M_1 = 3.0, \quad \nu_1 = 47.7334°$$

$$\nu_2 = \nu_1 + 15 = 62.7334°$$

Using the solver developed in Example 7.1, we obtain

$$M_2 = 4.0021$$

Now from the isentropic flow relations

$$\frac{T_1}{T_{o1}} = 0.3356 \qquad \frac{T_2}{T_{o2}} = 0.2211,$$

Since the flow is adiabatic, $T_{o1} = T_{o2}$ and therefore,

$$T_2 = \frac{T_2}{T_{o2}}\frac{T_{o2}}{T_{o1}}\frac{T_{o1}}{T_1}T_1 = (0.2211)(1)\left(\frac{1}{0.3356}\right)300 = 197.6460\,\mathrm{K}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{(1.44)(256)(197.6460)} = 269.9263\,\mathrm{m/s}$$

$V_2 = (M_2)(a_2) = (4.0021)(269.9263) = 1{,}080.2722$ m/s

Case (b) $\Delta = 30°$

$$M_1 = 3.0, \quad \nu_1 = 47.7334°$$

$$\nu_2 = \nu_1 + 30 = 77.7334°$$

Using the solver developed in Example 7.1, we obtain

$$M_2 = 5.6003$$

Now from the isentropic flow relations

$$\frac{T_1}{T_{o1}} = 0.3356 \qquad \frac{T_2}{T_{o2}} = 0.1266,$$

Since the flow is adiabatic, $T_{o1} = T_{o2}$ and therefore,

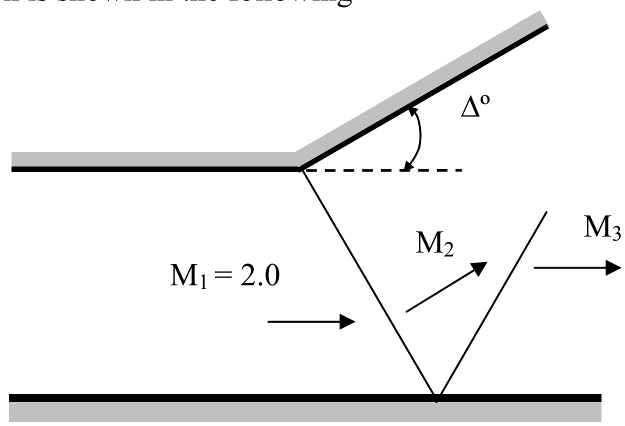
$$T_2 = \frac{T_2}{T_{o2}}\frac{T_{o2}}{T_{o1}}\frac{T_{o1}}{T_1}T_1 = (0.1266)(1)\left(\frac{1}{0.3356}\right)300 = 113.1704\,\mathrm{K}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{(1.44)(256)(113.1704)} = 204.2527\,\mathrm{m/s}$$

$V_2 = (M_2)(a_2) = (5.6003)(204.2527) = 1{,}143.8762$ m/s

Problem 17. – Air ($\gamma = 1.4$) at $M_1 = 2$ and $p_1 = 150$ kPa flows in a duct as shown in Figure 7.15. The upper wall turns the uniform supersonic stream through 5° “away” from the flow resulting in the formation of a Prandtl-Meyer fan at the corner. Waves of the fan reflect off the lower surface of the duct. Determine the Mach number and pressure downstream of the leading reflected expansion wave.

The flow configuration is shown in the following



$$M_1 = 2.0, \quad \nu_1 = 26.3798°$$

$$\nu_2 = \nu_1 + \Delta = 26.3798 + 5 = 31.3798°$$

Using the solver developed in Example 7.1, we obtain

$$M_2 = 2.1864,$$

Since the flow just downstream of the reflected leading wave was turned twice through the expansion, we may write

$$\nu_3 = \nu_2 + \Delta = \nu_1 + 2\Delta = 26.3798 + 10 = 36.3798°$$

So

$$M_3 = 2.3849,$$

Now from the isentropic relations at M_1 and M_3

$$\frac{p_1}{p_{o1}} = 0.126694 \qquad \frac{p_3}{p_{o3}} = 0.070239,$$

Hence,

$$p_3 = \frac{p_3}{p_{o3}}\frac{p_{o3}}{p_{o1}}\frac{p_{o1}}{p_1}p_1 = (0.070239)(1)\left(\frac{1}{0.126694}\right)150 = 83.1598\,\text{kPa}$$

Problem 18. – When Theodor Meyer presented his dissertation in 1908, the Mach number had not been named; it appeared 20 years later (see Ref. 2). Accordingly, at that time of Meyer's thesis the static to total pressure ratio was used. Write the Prandtl-Meyer function much like Meyer would have using the pressure ratio.

From Eq.(7.9) the Prandtl-Meyer function is written

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right] - \tan^{-1}\left(\sqrt{M^2-1}\right)$$

The static to total pressure relation is

$$\frac{p}{p_o} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

Therefore,

$$M^2 = \frac{2}{\gamma-1}\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} - \frac{2}{\gamma-1}$$

And so

$$M^2 - 1 = \frac{2}{\gamma-1}\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} - \frac{\gamma+1}{\gamma-1}$$

Replacing the Mach number in the Prandtl-Meyer expression brings

$$\nu = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{2}{\gamma+1}\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} - 1}\right] - \tan^{-1}\left(\sqrt{\frac{2}{\gamma-1}\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} - \frac{\gamma+1}{\gamma-1}}\right)$$

***Problem 19*. –** Obtain the following pressure-Mach number relation from the continuity and normal momentum equations applied to a control volume containing a Mach wave:

$$\frac{dp}{p} = -\frac{\gamma M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}$$

Integrate this relation to derive the expression for the pressure ratio across the Mach wave, p_2/p_1 in terms of M_1 and M_2, i.e., obtain Eq.(7.13).

From the normal momentum equation,

$$dp + \rho V dV = 0$$

Hence,

$$\frac{dp}{p} = -\gamma \frac{\rho}{\gamma p} V^2 \frac{dV}{V} = -\gamma \frac{V^2}{a^2} \frac{dV}{V} = -\gamma M^2 \frac{dV}{V}$$

But from Eq.(7.6)

$$\frac{dV}{V} = \left(\frac{1}{1 + \frac{\gamma - 1}{2} M^2} \right) \frac{dM}{M}$$

So that

$$\frac{dp}{p} = \left(\frac{-\gamma M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) \frac{dM}{M} = \frac{-\gamma M dM}{1 + \frac{\gamma - 1}{2} M^2}$$

If

$$f = 1 + \frac{\gamma - 1}{2} M^2$$

Take the logarithm and then differentiate to obtain

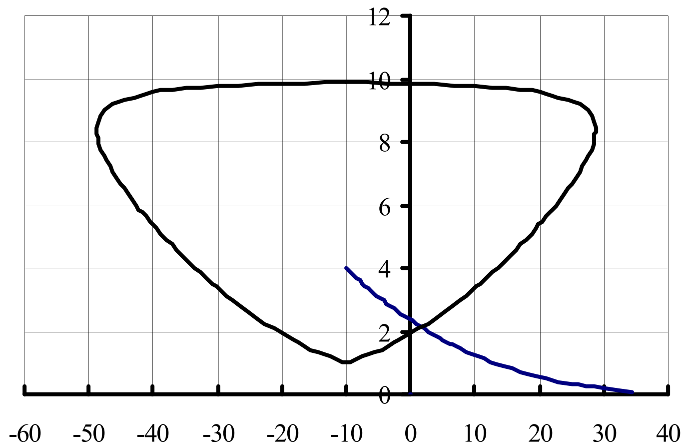
$$\frac{df}{f} = \frac{(\gamma - 1) M dM}{1 + \frac{\gamma - 1}{2} M^2} = -\frac{\gamma - 1}{\gamma} \frac{dp}{p}$$

Integration produces

$$\frac{p_2}{p_1} = \left(\frac{f_1}{f_2} \right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_2^2} \right)^{\gamma / (\gamma - 1)}$$

Problem 20. – Repeat Example 7.5 for $\gamma = 1.25$.

The pressure-flow direction diagram obtained for this flow is



The numerical solution for the intersection of the two curves is contained in the following table

iteration	y_{old}	$y+\Delta$	$y-\Delta$	f(y)	$f(y+\Delta)$	$f(y-\Delta)$	$\Delta f/\Delta y$	y_{new}	x	x (deg)
1	2.0000	2.0001	1.9999	-0.0380	-0.0380	-0.0380	0.2965	2.12816	0.20459	1.7223
2	2.1282	2.1283	2.1281	-0.0009	-0.0009	-0.0009	0.2826	2.13137	0.20504	1.7478
3	2.1314	2.1315	2.1313	0.0000	0.0000	0.0000	0.2823	2.13137	0.20504	1.7478
4	2.1314	2.1315	2.1313	0.0000	0.0000	0.0000	0.2823	2.13137	0.20504	1.7478

Thus,

$p_3/p_{ref} = p_4/p_{ref}$		$\alpha_3 = \alpha_4$	
2.1314		1.7478	
Expansion Region		**Shock Region**	
δ_{13}	11.7478	δ_{24}	11.7478
ν_3	41.7478	θ_{24}	28.2649
p_3/p_1	0.5328	p_4/p_2	2.1314
ρ_3/ρ_1	0.6043	ρ_4/ρ_2	1.8131
T_3/T_1	0.8817	T_4/T_2	1.1755
p_{o3}/p_{o1}	1.0000	p_{o4}/p_{o2}	0.9495
M_3	2.3686	M_4	2.5419

Chapter Eight

APPLICATIONS INVOLVING SHOCKS AND EXPANSION FANS

Problem 1. – A supersonic inlet (Figure P8.1) is to be designed to handle air (γ = 1.4, R = 287 J/kg·K) at Mach 1.75 with static pressure and temperature of 50 kPa and 250 K. Determine the diffuser inlet area A_i if the device is to handle 10 kg/s of air.

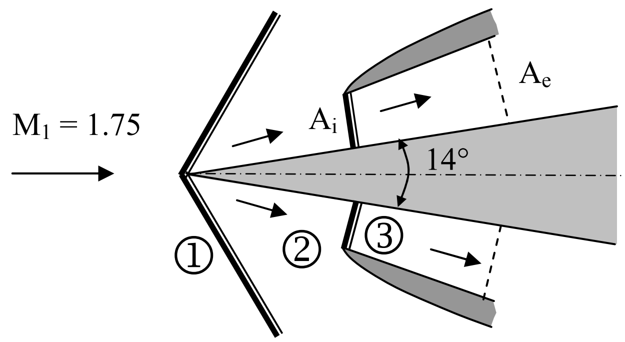


Figure P8.1

Using the oblique shock solution method we obtain

$$\left.\begin{aligned} M_1 &= 1.75 \\ \delta &= 7^\circ \end{aligned}\right\} \quad \begin{aligned} \theta &= 41.8715^\circ \\ M_2 &= 1.5090 \\ \frac{p_2}{p_1} &= 1.4251 \\ \frac{T_2}{T_1} &= 1.1079 \end{aligned}$$

$$p_2 = p_1\left(\frac{p_2}{p_1}\right) = 50(1.4251) = 71.2550 \text{ kPa}$$

$$T_2 = T_1\left(\frac{T_2}{T_1}\right) = 250(1.1079) = 276.9750 \text{ K}$$

Subscript 2 represents conditions just upstream of the normal shock

$$\dot{m} = \rho_2 A_2 V_2 = \left(\frac{p_2}{RT_2}\right) A_2 M_2 a_2 = \left(\frac{p_2}{RT_2}\right) A_2 M_2 \sqrt{\gamma RT_2} = 10\ \text{kg/s}$$

So

$$\left[\frac{71.2550}{0.287(276.9750)}\right] A_2 (1.5090)\sqrt{(1.4)(287)(276.9750)} = 10$$

$$A_2 = \frac{10\ \text{kg/s}}{(0.8964\ \text{kg/m}^3)(503.4015\ \text{m/s})} = 0.0222\ \text{m}^2$$

Problem 2. – The diffuser in Problem 1 is to further decelerate flow after the normal shock so that the velocity entering the compressor is not to exceed 25 m/s. Assuming isentropic flow after the shock, determine the area A_e required. For this condition, find the static pressure p_e. Take $\gamma = 1.4$ and $c_p = 1.004$ kJ/kg·K.

$$T_{oe} = T_{oi} = T_{o1} = \left(\frac{T_{o1}}{T_1}\right) T_1 = \left(\frac{1}{0.6202}\right) 250 = 403.0958\ \text{K}$$

For $M_2 = 1.5090$, the Mach number downstream of the normal shock is found to be $M_3 = 0.6979$. Hence, the area ratio for this Mach number can be obtained from the isentropic flow tables, $\frac{A_3}{A_3^*} = 1.0959$. And since the flow downstream of the normal shock is assumed to be isentropic $A_e^* = A_3^*$. Now

$$\frac{V_e^2}{2} + c_p T_e = c_p T_{oe}$$

$$T_e = T_{oe} - \frac{V_e^2}{2c_p} = 403.0958 - \frac{25^2\ \text{m}^2/\text{s}^2}{2(1.004\ \text{kJ/kg}\cdot\text{K})(1000\ \text{J/kJ})}$$
$$= 403.0958 - 0.3113 = 402.7845\ \text{K}$$

$$M_e = \frac{V_e}{a_e} = \frac{25}{\sqrt{1.4(287)(402.7845)}} = 0.0621$$

At this Mach number we can find $\frac{A_e}{A_e^*} = 9.3405$. Thus,

$$\frac{A_e}{A_i} = \frac{A_e}{A_e^*}\frac{A_e^*}{A_3^*}\frac{A_3^*}{A_3}\frac{A_3}{A_i} = (9.3405)(1)\left(\frac{1}{1.0959}\right)(1) = 8.5231$$

$$A_e = A_i\left(\frac{A_e}{A_i}\right) = 0.0222(8.5231) = 0.1892\ \mathrm{m}^2$$

Using the various Mach numbers that have been determined we can find the following corresponding pressure ratios from isentropic flow and normal shock relations

$$M_e = 0.0621, \frac{p_e}{p_{oe}} = 0.9973$$

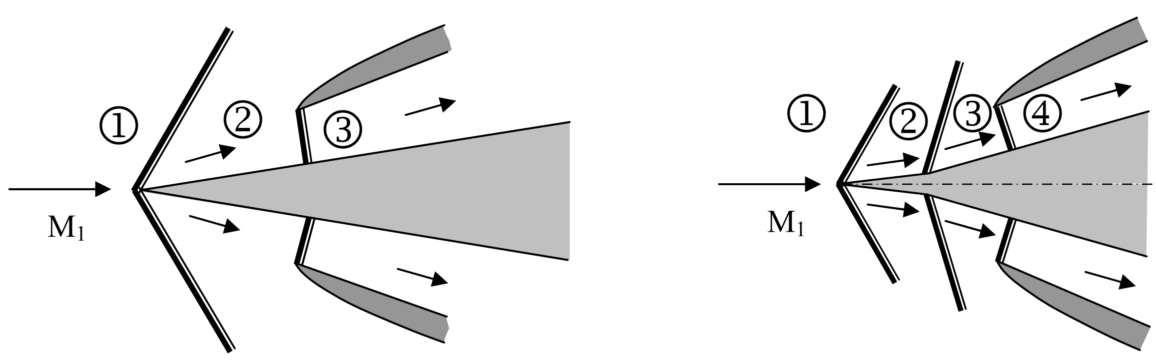
$$M_3 = 0.6979, \frac{p_3}{p_{o3}} = 0.7223$$

$$M_2 = 1.5090, \frac{p_3}{p_2} = 2.4899$$

Thus,

$$p_e = \frac{p_e}{p_{oe}}\frac{p_{oe}}{p_{o3}}\frac{p_{o3}}{p_3}\frac{p_3}{p_2}p_2 = (0.9973)(1)\left(\frac{1}{0.7223}\right)(2.4899)(71.5) = 245.8081\ \mathrm{kPa}$$

Problem 3. – Compare the loss in total pressure incurred by a one-shock spike diffuser with that incurred by a two-shock diffuser operating at Mach 2.0. Repeat at Mach 4.0 (see Figure 8.5). Assume that each oblique shock turns the flow through an angle of 10°. Take $\gamma = 1.3$.



(a) One-shock inlet (b) Two-shock inlet

Figure 8.5 Flow Regions within the Spike Diffusers of Example 8.1

From the oblique shock solver at $\gamma = 1.3$, $M_1 = 2.0$ and $\delta = 10°$, the weak solution yields $\theta = 38.8127°$. Moreover the Mach number downstream of the shock is, $M_2 = 1.6765$. For the one-shock diffuser,

$$\left(\frac{p_{o3}}{p_{o1}}\right)_{\text{one-shock}} = \left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right)$$

From the oblique shock relations at $M_1 = 2.0$, $p_{o2}/p_{ol} = 0.9861$ and from the normal shock relations at $M_2 = 1.6765$, $p_{o3}/p_{o2} = 0.8570$. Hence,

$$\left(\frac{p_{o3}}{p_{o1}}\right)_{\text{one-shock}} = (0.8570)(0.9861) = 0.8451$$

For the two-shock inlet, $M_2 = 1.6765$. At the latter Mach number and $\delta = 10°$, the wave angle for the weak shock solution is $\theta = 47.3152°$, $p_{o3}/p_{o2} = 0.9889$ and $M_3 = 1.3533$. At M_3 from the normal shock relations $p_{o4}/p_{o3} = 0.9677$. Thus,

$$\left(\frac{p_{o4}}{p_{o1}}\right)_{\text{two-shocks}} = \left(\frac{p_{o4}}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right) = (0.967)(0.9889)(0.9861) = 0.9437$$

Now at $M_1 = 4.0$ and $\delta = 10°$, the weak solution yields $\theta = 21.8411°$, $p_{o2}/p_{ol} = 0.9301$ and $M_2 = 3.4050$. From the normal shock relations at $M_2 = 3.4050$, $p_{o3}/p_{o2} = 0.1853$. Therefore, for the one oblique shock diffuser,

$$\left(\frac{p_{o3}}{p_{o1}}\right)_{\text{one-shock}} = \left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right) = (0.1853)(0.9301) = 0.1723$$

For the two-shock inlet, $M_2 = 3.4050$. At $M_2 = 3.4050$ and $\delta = 10°$, $\theta = 24.4808°$, $p_{o3}/p_{o2} = 0.9533$ and $M_3 = 2.9186$. Using M_3 in the normal shock relations gives $p_{o4}/p_{o3} = 0.3065$. For this case,

$$\left(\frac{p_{o4}}{p_{o1}}\right)_{\text{two-shocks}} = \left(\frac{p_{o4}}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right) = (0.3065)(0.9533)(0.9301) = 0.2718$$

Problem 4. – A *converging nozzle* is supplied from a large air ($\gamma = 1.4$, $R = 287$ J/kg·K) reservoir maintained at 600K and 2 MPa. If the nozzle back pressure is 101 kPa, determine the pressure and Mach number that exist at the nozzle exit plane. Since the nozzle is operating in the underexpanded regime, expansion waves occur at the nozzle exit. Determine the flow direction after the initial expansion fans and the flow Mach number.

Since the nozzle is operating in the underexpanded flow regime, the nozzle is choked. Accordingly, the Mach number at the exit is $M_e = 1.0$ and the exit pressure to reservoir pressure ratio is $p_e/p_o = 0.5283$ for $\gamma = 1.4$. Thus the exit pressure is

$$p_e = 0.5283(2\text{ MPa}) = 1056.6\text{ kPa}$$

The expansion fans turn the supersonic flow and reduce the pressure to that of the back pressure. Now

$$\frac{p_b}{p_o} = \frac{p_b}{p_e}\frac{p_e}{p_o} = \frac{101}{1056.6}0.5283 = 0.0505$$

From this pressure ratio we can find the corresponding Mach number

$$M_b = 2.5951$$

Since the flow expands from $M_e = 1$ to $M_b = 2.5951$, we need only determine the Prandlt-Meyer function for the latter Mach number. This provides the angle through which the flow is turned, i.e., $\nu_b - \nu_e = 41.3044 - 0 = 41.3044° = \alpha_b - \alpha_e = \alpha_b$

Problem 5. – An oblique shock wave occurs in a supersonic flow in which $M_1 = 3$. The shock turns the supersonic stream through 10°. The shock impinges on a free surface along which the pressure is constant and equal to p_1, i.e., the pressure upstream of the shock. The shock is reflected from the free surface as an expansion fan. Determine the Mach number and the angle of the flow just downstream of the fan. Assume $\gamma = 1.4$.

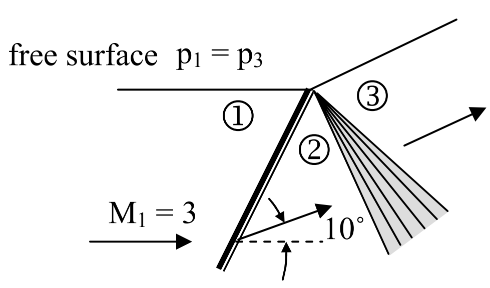


Figure P8.5

Using the oblique shock solution method we obtain

$$\left.\begin{array}{r} M_1 = 3.0 \\ \delta = 10° \end{array}\right\} \begin{array}{l} \theta = 27.3827° \\ M_2 = 2.5050 \\ \dfrac{p_2}{p_1} = 2.0545 \end{array}$$

In region 2 from the Prandtl-Meyer and isentropic relations

$$M_2 = 2.5050,\ \nu_2 = 39.2402° \text{ and } \frac{p_2}{p_{o2}} = 0.05807$$

Because the flow across the expansion fan is isentropic $p_{o2} = p_{o3}$ and because of the constant pressure free surface $p_1 = p_3$, thus we may form the following string of pressure ratios

$$\frac{p_3}{p_{o3}} = \frac{p_3}{p_1}\frac{p_1}{p_2}\frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o3}} = (1)\left(\frac{1}{2.0545}\right)(0.05807)(1) = 0.028265$$

With this pressure ratio, using the static to total pressure-Mach number relation, we obtain $M_3 = 2.9750$ and therefore from the Prandtl-Meyer relation $\nu_3 = 49.2727°$. Finally then, for this flow geometry

$$\nu_3 - \nu_2 = \alpha_3 - \alpha_2 = 49.2727 - 39.2402 = 10.0325°$$

Accordingly,

$$\alpha_3 = \alpha_2 + 10.0325 = 10 + 10.0325 = 20.0325°$$

Problem 6. – A *converging-diverging* nozzle is designed to provide exit flow at Mach 2.2. With the nozzle exhausting to a back pressure of 101 kPa, however, and a reservoir pressure of 350 kPa, the nozzle is overexpanded, with oblique shocks appearing at the exit. Determine the flow direction, static pressure, and Mach number in regions ①,②, and ③ of Figure P8.6.

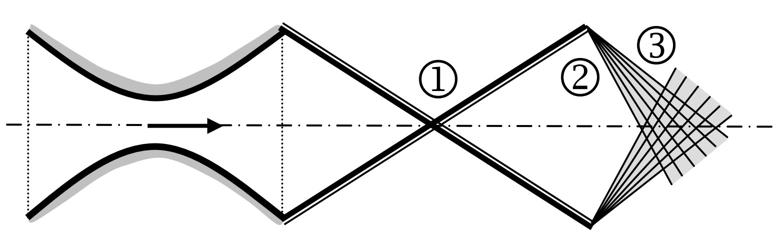
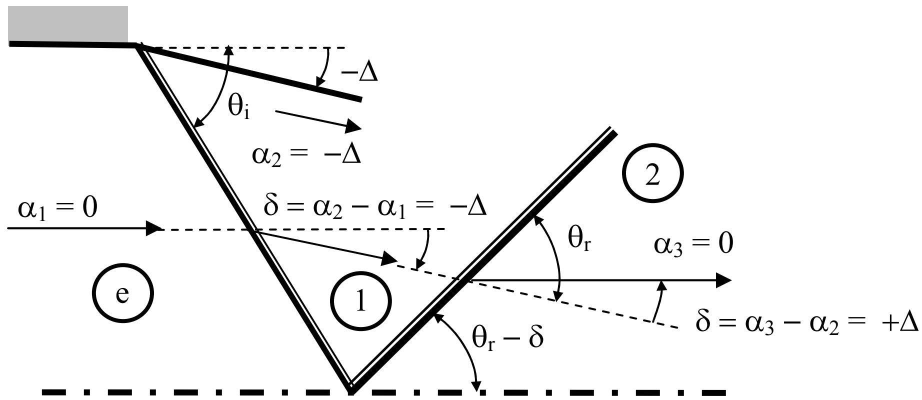


Figure P8.6

Consider the following geometry



From the isentropic flow relations at $M_e = 2.2$

$$\frac{p_e}{p_r} = \frac{p_e}{p_o} = 0.0935$$

Thus,

$$p_e = 0.0935(350 \text{ kPa}) = 32.7250 \text{ kPa}$$

$$\frac{p_1}{p_e} = \frac{101}{32.7250} = 3.0863$$

From this pressure ratio, essentially p_2/p_1, and the normal shock pressure-Mach number relation, we can determine the upstream normal component to the oblique shock as

$$M_{ne} = 1.6698$$

And since the ratio of M_{ne} to M_e is the $\sin\theta$ we can therefore determine the shock wave angle

$$\frac{M_{ne}}{M_e} = \frac{1.6698}{2.20} = 0.7590 = \sin\theta, \quad \text{hence} \quad \theta = -49.3761°$$

With M_e and θ, we can find the deflection angle to be $\delta = -20.8875°$. Accordingly, the flow in ① is turned 20.8875° from the horizontal. Moreover, the Mach number in this region is $M_1 = 1.3596$.

At this Mach number and for a flow deflection of +20.8875°, there is no solution (see Figure 6.6). Reflection must be as in Figure 6.14, i.e., a Mach reflection will occur.

***Problem** 7*. – Determine the flow directions in regions ① and ③ of Figure P8.6 if the reservoir pressure were increased to 2 MPa.

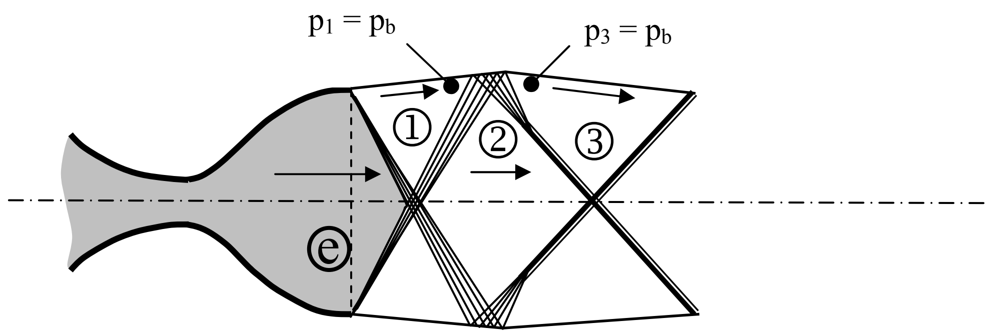
Because the nozzle is designed for an exit Mach number, M_e, of 2.2, it follows that the static to total pressure ratio at the nozzle exit is

$$\frac{p_e}{p_o} = 0.0935$$

The back pressure is 101 kPa and the reservoir pressure is 2,000 kPa, therefore,

$$\frac{p_b}{p_o} = \frac{101}{2000} = 0.0505 \quad .$$

Since $p_b/p_o < p_e/p_o$, the nozzle is underexpanded for this back pressure-reservoir pressure combination. The following provides nomenclature and a sketch of the flow field.



From the Prandtl-Meyer relation at $M_e = 2.2$, we find that $\nu_e = 31.7325^\circ$. Since $p_b = p_1$ in region 1. Then, from the static to total pressure ratio, $p_1/p_o = 0.0505$, we find that $M_1 = 2.5950$ and therefore $\nu_1 = 41.3044^\circ$. So the exit flow is turned through the following angle as it passes into region 1

$$\alpha_1 - \alpha_e = \nu_1 - \nu_e = 41.3044 - 31.7325 = 9.5719^\circ$$

Since $\alpha_e = 0^\circ$, then it follows that $\alpha_1 = 9.5719^\circ$. The flow in region 2 must be horizontal, i.e., $\alpha_2 = 0^\circ$, and since we must pass through another expansion fan, we may write that

$$-(\alpha_2 - \alpha_1) = \nu_2 - \nu_1$$

Hence,

$$\nu_2 = \nu_1 + \alpha_1 = 41.3044 + 9.5719 = 50.8763^\circ$$

From this we find $M_2 = 3.0587$ and therefore $p_2/p_{o2} = 0.0249$. Since the flow is isentropic across both expansion fans, $p_{o2} = p_{o1} = p_o = 2{,}000$ kPa. This enables us to determine $p_2 = (0.0249)(2{,}000) = 49.8000$ kPa. Now since $p_3 = p_b = 101$ kPa, we have the pressure ratio across the oblique shock, i.e., p_3/p_2 = 101/49.8 = 2.0281. With this pressure ratio and M_2 = 3.0587 we can first determine the shock angle from Eq.(6.10)

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\theta}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$$

Hence

$$\sin\theta = \pm\sqrt{\frac{(2.0281)(2.4)+0.4}{(2.8)(3.0587)^2}} = \pm 0.4484$$

From which we find that $\theta = \pm 26.6423^\circ$. With this angle and the Mach number we enter the oblique shock solver to find $\delta = \pm 9.6362^\circ$. From this we see that $\alpha_3 = -9.6362^\circ$.

Problem 8. – A plug nozzle is designed to produce Mach 2.5 flow in the axial direction at the plug apex. Flow at the throat cowling must therefore be directed toward the axis. Determine the flow direction at the throat cowling to produce axial flow at the apex. Assume γ = 1.4.

$$M_{th} = 1 \text{ to } M_{ap} = 2.5, \quad {}_{ap} - \nu_{th} = 39.1236 - 0 = 39.1236^\circ = \alpha_{ap} - \alpha_{th} = 0 - \alpha_{th}$$

Problem 9. – A rocket nozzle is designed to operate with a ratio of chamber pressure to ambient pressure (p_c/p_a) of 50. Compare the performance of a plug nozzle with that of a converging-diverging nozzle for two cases where the nozzle is operating overexpanded; p_c/p_a = 40 and p_c/p_a = 20. Compare on the basis of thrust coefficient; $C_T = \mathbf{T}/(p_c A_{th})$, where $\mathbf{T}$ is the thrust and A_{th} is the area of throat. Assume γ = 1.3 and in both cases neglect the effect of nonaxial exit velocity components.

For the design case,
From $p_e/p_o = p_a/p_c$ = 1/50 = 0.02, since in the design case the flow is isentropic, we can determine the Mach number at the exit, i.e., M_e = 3.1267 (see Eq. (3.15)), and therefore $T_e/T_o = T_e/T_c$ = 0.4054. Now from the definition of the thrust coefficient

$$C_T = \frac{(\dot{m}_{th} V_e)}{p_c A_{th}} = \frac{(\rho_{th} A_{th} V_{th}) V_e}{p_c A_{th}} = \left(\frac{p_{th}}{RT_{th}}\right)\frac{V_{th} V_e}{p_c} = \left(\frac{p_{th}}{p_c}\right)\left(\frac{p_c}{RT_o}\right)\left(\frac{T_o}{T_{th}}\right)\frac{(M_{th} a_{th})(M_e a_e)}{p_c}$$

Because the nozzle is choked, M_{th} = 1 and for γ = 1.3,

$$\frac{p_{th}}{p_c} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{1-\gamma}} = 0.5457$$

$$\frac{T_{th}}{T_c} = \frac{2}{\gamma+1} = 0.8696$$

Using these values and the values at the exit, we get

$$(C_T)_{design} = \left[\frac{0.5457 p_c}{R(0.8696 T_c)}\right]\left[\frac{\sqrt{(1.3)(R)(0.8696)T_c}}{p_c}\right](3.1267)\sqrt{(1.3)(R)(0.4054)T_c}$$

$$= 1.5145$$

Note, R, p_c and T_c drop out of the above expression.

For the converging-diverging nozzle operating off design,

$$C_T = (C_T)_{design} + \frac{A_e(p_e - p_a)}{A_{th} p_c} = 1.5145 + \frac{A_e}{A_{th}}\left(\frac{p_e}{p_c} - \frac{p_a}{p_c}\right)$$

where at $M_e = 3.1267$, $A_e/A_{th} = A_e/A^* = 5.9590$. So for $p_c/p_a = 40$,

$$C_T = 1.5145 + 5.9590\left(\frac{1}{50} - \frac{1}{40}\right) = 1.4847$$

And for $p_c/p_a = 20$,

$$C_T = 1.5145 + 5.9590(0.02 - 0.05) = 1.3357$$

For the plug nozzle,
Flow in the plug nozzle does not continue to expand below ambient pressure, so there is no pressure term in the expression for thrust.

Now at $\frac{p_c}{p_a} = 40, \quad M_e = 2.9918, \quad \text{and} \quad \frac{T_e}{T_c} = 0.4269$

$$C_T = \frac{\dot{m}_{th} V_e}{p_c A_{th}} = \left[\frac{0.5457 p_c}{R(0.8696 T_c)}\right]\left[\frac{A_{th}\sqrt{1.3(R)0.8696 T_c}}{p_c A_{th}}\right] 2.9918\sqrt{1.3R(0.4269 T_c)}$$

$$= 1.4871$$

Whereas for $p_c/p_a = 20$, $M_e = 2.5773$ and $\frac{T_e}{T_c} = 0.5009$

$$C_T = 1.3876$$

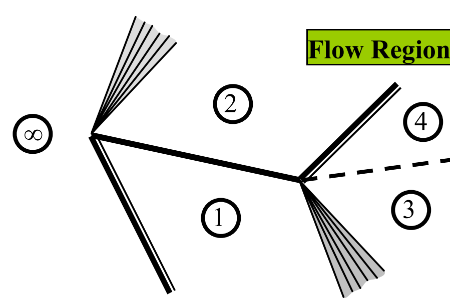
When these results are compared to those of Example 8.4, it is seen that the effect of changing γ from 1.4 to 1.3 is relatively small.

Problem 10. – Compute the lift and drag coefficients for a flat plate airfoil of chord length c = 1m in supersonic flow through air ($\gamma = 1.4$) at $M_\infty = 3$ and $\alpha = 8°$.

Because this is a companion to Example 8.5 rather than repeat the same format, instead the results of the spreadsheet program for a flat plate at an angle of attack are presented.

Input Parameters	
M_∞	3.0000
α	8.0
γ	1.4
c	1.0

Flow Regions



Results

Region ∞: freestream

γ	M_∞	α_∞	ν_∞	$\rho v^2/(2p_\infty)$		$p_\infty/p_{o\infty}$	
1.4	3.0	0	49.7573	6.3000		0.02722	

Region 1: lower region behind oblique shock

γ	M_∞	δ	θ	α	M_1	$p_{o1}/p_{o\infty}$	p_1/p_∞
1.4	3.0	8.0	25.6114	8.0	2.6031	0.9799	1.7953

Region 2: upper region behind expansion fan

γ	M_∞	ν_∞	α	ν_2	M_2	p_2/p_{o2}	p_2/p_∞
1.4	3.0	49.7573	8.0	57.7573	3.4519	0.01404	0.51574

Region 3: lower region behind expansion fan							
γ	M_1	δ_{13}		ν_3	M_3	p_{o3}/p_{o1}	p_3/p_1
1.4	2.6031	8.0233		49.5077	2.9871	1.0000	0.5565

Region 4: upper region behind oblique shock							
γ	M_2	δ_{24}	θ_{24}		M_4	p_{o4}/p_{o2}	p_4/p_2
1.4	3.4519	8.0233	22.8937		2.9812	0.9712	1.9372

Direction of trailing flow
$\alpha_3 = \alpha_4$
0.0233

Lift and Drag Coefficents	
C_L	C_D
0.2011	0.0283
$L/(cp_\infty)$	$D/(cp_\infty)$
1.2671	0.1781

Mach numbers and pressure ratios				
M_∞	M_1	M_2	M_3	M_4
3.0	2.6031	3.4519	2.9871	2.9812
p_∞/p_∞	p_1/p_∞	p_2/p_∞	p_3/p_∞	p_4/p_∞
1.0000	1.7953	0.5157	0.9991	0.9991

Problem 11. – Compute the drag coefficient for a symmetric, diamond-shaped airfoil (Figure P8.11) with a thickness to chord ratio, t/c, equal to 0.10 flying at Mach 3.5 in air ($\gamma = 1.4$) at 10 km at zero angle of attack.

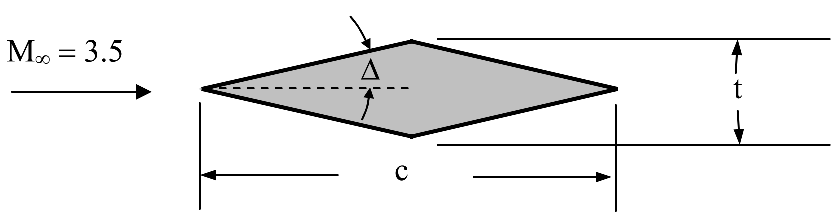


Figure P8.11

For an oblique shock at the nose of the airfoil,

$$\tan\Delta = \frac{t}{c} = 0.10, \quad \Delta = \delta = 5.7106°$$

$$\left.\begin{array}{l} M_\infty = 3.5 \\ \delta = 5.7106° \end{array}\right\} \begin{array}{l} \theta = 20.7409°, \\ \dfrac{p_1}{p_\infty} = 1.6257, \quad M_1 = 3.1566, \quad \nu_1 = 52.6880, \quad \dfrac{p_1}{p_{o1}} = 0.0216 \end{array}$$

$$\nu_3 = \nu_1 + 2\Delta = 52.6880 + 11.4212 = 64.1092°$$

Using this value of the Prandtl-Meyer function we find the corresponding Mach number is $M_3 = 3.8760$ and in turn the corresponding static to total pressure ratio is $p_3/p_{o3} = 0.007781$. Accordingly, we may form the following ratio

$$\frac{p_3}{p_2} = \frac{p_3}{p_{o3}} \frac{p_{o3}}{p_{o1}} \frac{p_{o1}}{p_1} = (0.007781)(1)\frac{1}{0.02156} = 0.3609$$

Therefore,

$$p_2 = 1.6257 p_\infty, \quad p_3 = \frac{p_3}{p_2} p_2 = (0.3609)(1.6257 p_\infty) = 0.5867 p_\infty$$

Because of the symmetry and the 0º angle of attack, the lift coefficient is zero. The drag coefficient may be determined in the following way

$$D = (p_2 - p_3)t$$

$$C_D = \frac{(p_2 - p_3)t}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} = \frac{(1.6257 - 0.5867)}{\frac{1}{2}(1.4)(3.5)^2}(0.1) = 0.0121$$

Problem 12. Compute the lift and drag coefficients for the airfoil described in Problem 11 for an angle of attack of 5°.

Upper Surface

$$M_\infty = 3.5, \quad \nu_\infty = 58.5298°$$

$$\nu_2 = \nu_\infty - \alpha_2 = 58.5298 - (0.7106) = 59.2404°$$

$$M_2 = 3.5450$$

$$\nu_4 = \nu_2 + \alpha_2 - \alpha_4 = 59.2404 + (0.7106) - (-10.7106) = 70.6616°$$

$$M_4 = 4.3961$$

$$\frac{p_4}{p_2} = \frac{p_4}{p_{o4}} \frac{p_{o4}}{p_{o2}} \frac{p_{o2}}{p_2} = (0.003937)(1)\left(\frac{1}{0.003937}\right) = 0.3201, \quad \frac{p_2}{p_\infty} = 9382$$

Lower Surface

$$\left.\begin{array}{l} M_\infty = 3.5, \\ \delta_{\infty 1} = \alpha + \Delta = 5 + 5.7106 = 10.7106° \end{array}\right\} \begin{array}{l} \theta = 25.0309°, \\ M_1 = 2.8623, \quad \nu_1 = 47.0292° \end{array}$$

$$\nu_3 = \nu_2 + 2\Delta = 47.0292 + 11.4212 = 58.4504°, \; M_3 = 3.4950$$

$$\frac{p_3}{p_1} = \frac{p_3}{p_{o3}}\frac{p_{o3}}{p_{o1}}\frac{p_{o1}}{p_1} = (0.01320)(1)\left(\frac{1}{0.03351}\right) = 0.3939, \quad \frac{p_1}{p_\infty} = 2.3918$$

$$C_L = \frac{\left[\dfrac{p_1 \frac{c}{2}\cos(10.7106)}{\cos(5.7106)} + \dfrac{p_2 \frac{c}{2}\cos(0.7106)}{\cos(5.7106)}\right] - \left[\dfrac{p_3 \frac{c}{2}\cos(0.7106)}{\cos(5.7106)} + \dfrac{p_4 \frac{c}{2}\cos(10.7106)}{\cos(5.7106)}\right]}{\left(\frac{1}{2}\gamma p_\infty M_\infty^2 c\right)}$$

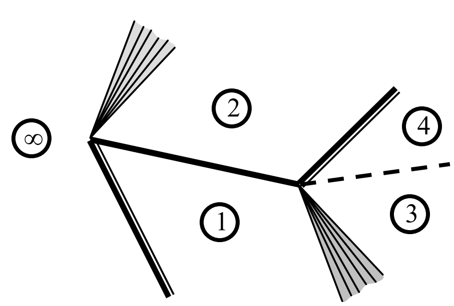
$$C_L = \frac{(2.3918 - 0.3003)(0.9826) + (0.9382 - 0.9424)(0.9999)}{(0.9950)(1.4)(3.5)^2} = 0.1202$$

$$C_D = \frac{p_1 \sin(10.7106) + p_2 \sin(0.7106) - p_3 \sin(0.7106) - p_4 \sin(10.7106)}{p_\infty \cos(5.7106)(1.4)(3.5)^2}$$

$$C_D = \frac{(2.3918 - 0.3003)(0.1858) + (0.9382 - 0.9424)(0.0124)}{(0.9950)(1.4)(3.5)^2} = 0.0228$$

Problem 13. – Compare the lift to drag ratio of the diamond airfoil in problem 12 with that of a flat-plate airfoil for the same freestream Mach number of 3.5 and angle of attack of 5°. Assume $\gamma = 1.4$.

Flat plate airfoil
The various flow regions are numbered as follows



Upper Surface

Using the freestream Mach number

$$\nu_\infty = 58.5298°$$

From this and the angle of attack we can find ν_2 from which we can find M_2

$$\nu_\infty + \alpha_\infty = \nu_2 + \alpha_2$$
$$\therefore \nu_2 = \nu_\infty + \alpha_\infty - \alpha_2 = 58.5298 + 0 - (-5.0000) = 63.5298°$$

and so $M_2 = 3.8344$. Furthermore using the Mach of the freestream and in region 2 we can use the isentropic relations to determine the corresponding static to stagnation pressure ratios. Since the flow from the freestream into region 2 is isentropic

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o\infty}}\frac{p_{o\infty}}{p_\infty} = (0.008233)(1.0)\left(\frac{1}{0.013111}\right) = 0.6280$$

Lower Surface

Because the freestream flow must be turned through 5º as it passes through the oblique shock

$$\left.\begin{array}{l} M_\infty = 3.5 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta = 20.1813°, \\ M_1 = 3.1983 \quad \dfrac{p_1}{p_\infty} = 1.5343 \end{array}$$

$$C_L = \frac{(p_1 - p_2)(c)\cos(\alpha)}{\left(\frac{1}{2}\gamma p_\infty M_\infty^2 c\right)} = \frac{(1.5343 - 0.6280)\cos(5.0)}{\frac{1}{2}(1.4)(3.5)^2} = \frac{(0.9063)(0.9962)}{(8.5750)} = 0.1053$$

$$C_D = C_L \tan(\alpha) = (0.1053)(0.08749) = 0.00921$$

$$\left(\frac{C_L}{C_D}\right)_{\text{flat plate}} = \frac{0.1053}{0.00921} = 11.4301 \text{ at } 5°, \left(\frac{C_L}{C_D}\right)_{\text{diamond foil}} = \frac{0.1202}{0.0228} = 5.2719 \text{ at } 5°$$

Problem 14. – Consider a flat-plate supersonic airfoil with a flap, as shown in Figure P8.10. For a flap angle of 5°, an angle of attack 10°, and a flight Mach number of 2.2, find the lift and drag coefficients of the airfoil.

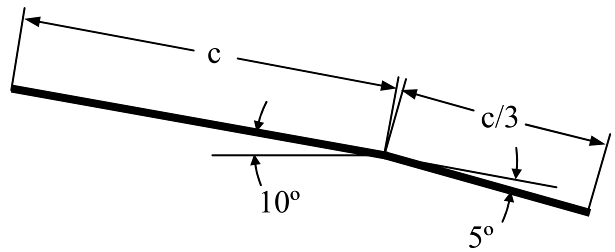
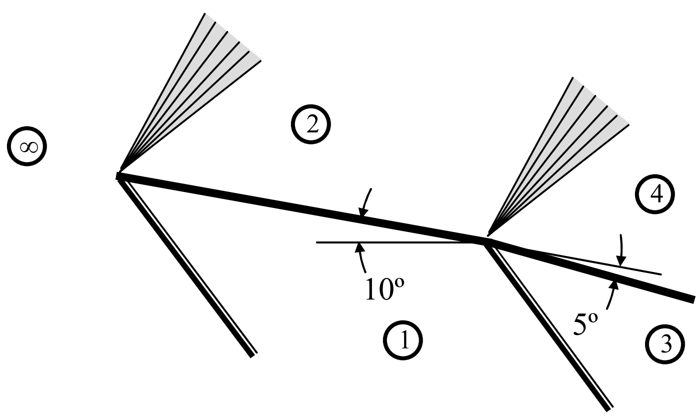


Figure P8.14

Except for the trailing edge phenomena, there will be two expansion fans on the top of the plate and two oblique shocks on the lower portion. The regions for the calculations are numbered as follows



From this and the angle of attack we can find ν_2, which will lead to M_2

$$\nu_\infty + \alpha_\infty = \nu_2 + \alpha_2$$
$$\therefore \nu_2 = \nu_\infty + \alpha_\infty - \alpha_2 = 31.7325 + 0 - (-10.0000) = 41.7325°$$

and so M_2 = 2.6142. This process is repeated in passing through the expansion fan at the corner of the flat plate and the flap

$$\nu_2 + \alpha_2 = \nu_4 + \alpha_4$$
$$\therefore \nu_4 = \nu_2 + \alpha_2 - \alpha_4 = 41.7325 - 10.0000 - (-15.0000) = 46.7325°$$

and so M_4 = 2.8478. Furthermore using M_∞, M_2, M_4, we can use the isentropic relations to determine the corresponding static to stagnation pressure ratios. Since the flows from the freestream into region 2 and from region 2 to 4 are isentropic

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_{o2}} \frac{p_{o2}}{p_{o\infty}} \frac{p_{o\infty}}{p_\infty} = (0.04903)(1.0)\left(\frac{1}{0.09352}\right) = 0.5242$$

$$\frac{p_4}{p_\infty} = \frac{p_4}{p_{o4}} \frac{p_{o4}}{p_{o2}} \frac{p_{o2}}{p_{o\infty}} \frac{p_{o\infty}}{p_\infty} = (0.03426)(1.0)(1.0)\left(\frac{1}{0.09352}\right) = 0.3663$$

Lower Surface

The freestream flow is turned through 10º as it passes through the first oblique shock. Therefore,

$$\left.\begin{matrix} M_\infty = 2.2 \\ \delta = 10° \end{matrix}\right\} \begin{matrix} \theta = 35.7855°, \\ M_1 = 1.8228 \quad \dfrac{p_1}{p_\infty} = 1.7641 \end{matrix}$$

The stream in region 1 is turned through 5º as it passes through the second oblique shock as it flows into region 3. Therefore,

$$\left.\begin{matrix} M_1 = 1.8228 \\ \delta = 5° \end{matrix}\right\} \begin{matrix} \theta = 37.9098°, \\ M_3 = 1.6502 \quad \dfrac{p_3}{p_1} = 1.2967 \end{matrix}$$

And so

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_1} \frac{p_1}{p_\infty} = (1.2967)(1.7641) = 2.2875$$

$$\begin{aligned} C_L &= \frac{(p_1 - p_2)c\cos(10)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} + \frac{(p_3 - p_4)(c/3)\cos(15)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} \\ &= \frac{(1.7641 - 0.5242)(0.9848)}{(3.3880)} + \frac{(2.2875 - 0.3663)(0.9659)}{(3.3880)(3)} \\ &= 0.3604 + 0.1826 = 0.5430 \end{aligned}$$

$$C_D = \frac{(p_1 - p_2)c\sin(10)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} + \frac{(p_3 - p_4)(c/3)\sin(15)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c}$$

$$= \frac{(p_1 - p_2)c\cos(10)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c}\tan(10) + \frac{(p_3 - p_4)(c/3)\cos(15)}{\frac{1}{2}\gamma p_\infty M_\infty^2 c}\tan(15)$$

$$= (0.3604)(0.1763) + (0.1826)(0.2679) = 0.0635 + 0.0489$$

$$= 0.1125$$

Problem 15. – Compute the lift and drag coefficients for the supersonic, symmetric airfoil shown flying in air (γ = 1.4) at Mach 2.5 at an angle of attack of 5° in Figure P8.15.

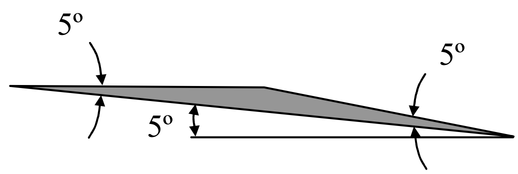


Figure P8.l5

Because the angle of attack and the wedge angle have the same value, the flow will experience only one expansion fan on the upper surface where the slope changes and an oblique shock on the bottom at the leading edge.

Upper Surface

$$\nu_\infty + \alpha_\infty = \nu_2 + \alpha_2$$

$$\therefore \nu_2 = \nu_\infty + \alpha_\infty - \alpha_2 = 39.1236 + 0 - (-10.0000) = 49.1236°$$

and so M_2 = 2.9674.

$$\frac{p_2}{p_\infty} = \frac{p_2}{p_{o2}}\frac{p_{o2}}{p_{o\infty}}\frac{p_{o\infty}}{p_\infty} = (0.02859)(1.0)\left(\frac{1}{0.05853}\right) = 0.4885$$

Lower Surface

The freestream flow is turned through 5º as it passes through the forward oblique shock. Therefore,

$$\left.\begin{array}{l} M_\infty = 2.5 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta = 27.4227°, \\ M_1 = 2.2915 \quad \dfrac{p_1}{p_\infty} = 1.3799 \end{array}$$

$$C_L = \frac{p_1 c \cos(5) - \dfrac{p_\infty c}{2\ \cos(5)} - \dfrac{p_2 c \cos(10)}{2\ \cos(5)}}{\dfrac{1}{2}\gamma p_\infty M_\infty^2 c}$$

$$= \frac{(1.3799)(0.9962) - \dfrac{0.5}{(0.9962.)} - \dfrac{(0.4885)(0.9848)}{2(0.9962)}}{(4.3750)} = \frac{1.3747 - 0.5019 - 0.2415}{4.3750}$$

$$= 0.1443$$

$$C_D = \frac{p_1 c \sin(5) - \dfrac{p_2 c \sin(10)}{2\ \cos\ 5°}}{\dfrac{1}{2}\gamma p_\infty M_\infty^2 c}$$

$$= \frac{(1.3747)(0.0875) - (0.2415)(0.1763)}{4.3750} = 0.0178$$

Problem 16. – A supersonic jet plane is flying horizontally at 150 m above ground level at a Mach number of 2.5, as shown in Figure P8.16. The airfoil is symmetric and diamond shaped, with 2Δ = 10º and a chord length of 4m. As the plane passes over, a ground observer hears the "sonic boom" caused by the shock waves. Find the time between the two "booms," one from the shock at the leading edge and one from the shock at the trailing edge. Ambient pressure and temperature are 100 kPa and 20°C.

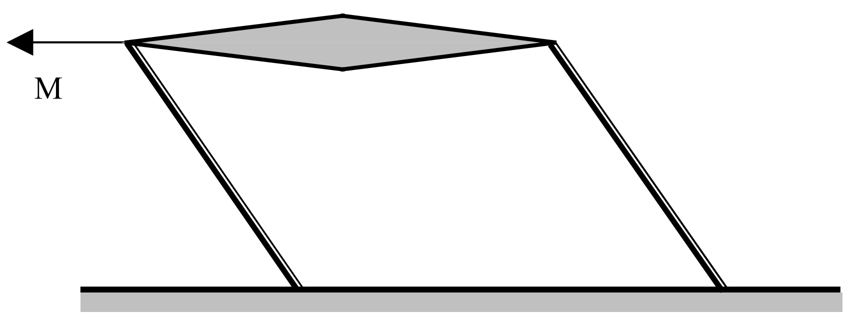
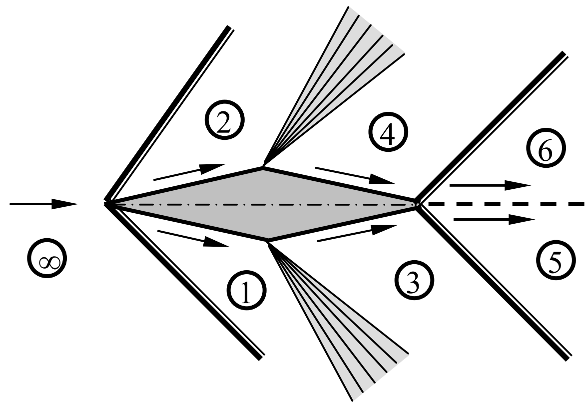


Figure P8.l6

The region numbering is shown in the following figure



Only the lower surface need be considered. For the first shock wave

$$\left.\begin{array}{l} M_\infty = 2.5 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta_{\infty-1} = 27.4227°, \\ M_1 = 2.2915 \end{array}$$

For the expansion fan

$$M_1 = 2.2915 \} \begin{array}{l} \nu_1 = 34.0700° \\ \nu_3 = 34.0700 + 10 = 44.0700° \\ M_3 = 2.7208 \end{array}$$

Finally for the second oblique shock wave,

$$\left.\begin{array}{l} M_3 = 2.7208 \\ \delta = 5° \end{array}\right\} \begin{array}{l} \theta_{1-3} = 25.3093°, \\ M_5 = 2.4951 \end{array}$$

If the airfoil is H above the surface and if the distance between shocks at the surface is called D. Then

$$D = c + H\cot(\theta_{1-2} - \Delta) - H\cot(\theta_{\infty-1}) = 4 + \frac{150}{\tan(20.3093)} - \frac{150}{\tan(27.4227)}$$

$$= 4 + 403.3005 - 289.0989 = 120.2016\text{m}$$

$$\Delta t = \frac{D}{V_\infty} = \frac{D}{M_\infty a_\infty} = \frac{120.2016}{(2.5)\sqrt{(1.4)(287)(293)}} = 0.1401\text{s}$$

Chapter Nine

FLOW WITH FRICTION

Problem 1. – Draw the T-s diagram for the adiabatic flow of a gas with $\gamma = 1.4$ in a constant diameter pipe with friction. The reference Mach number, M_1, for the flow is 3.0.

Following Example 9.1, $\dfrac{T_o}{T_1} = 1 + \dfrac{\gamma - 1}{2} M_1^2 = 1 + \left(\dfrac{1.4 - 1}{2}\right)(3)^2 = 2.8000$

Equation (9.7) may now be written as

$$\frac{\Delta s}{c_p} = \frac{1}{\gamma}\ln\left(\frac{T}{T_1}\right) + \frac{\gamma - 1}{2\gamma}\ln\left(\frac{T_o - T}{T_o - T_1}\right) = \ln\left\{\left(\frac{T_o/T_1}{T_o/T}\right)^{\frac{1}{\gamma}}\left[\left(\frac{T_o/T_1}{T_o/T}\right)\left(\frac{T_o/T - 1}{T_o/T_1 - 1}\right)\right]^{\frac{\gamma-1}{2\gamma}}\right\}$$

In this expression there are two values of T_o/T that will cause $\Delta s/c_p$ to vanish. Clearly, both will cause the argument of the natural log function to be exactly equal to 1. One value occurs at T_o/T_1, i.e., when $T = T_1$. Because of the nonlinearity of the function involving T_o/T, the other value must be found numerically. This is readily accomplished using a spreadsheet program to implement the Newton-Raphson method. Setting the argument of the natural log function to unity gives

$$\left(\frac{T_o/T_1}{T_o/T}\right)^{\frac{1}{\gamma}}\left[\left(\frac{T_o/T_1}{T_o/T}\right)\left(\frac{T_o/T - 1}{T_o/T_1 - 1}\right)\right]^{\frac{\gamma-1}{2\gamma}} = 1$$

Rearranging this produces

$$f\left(\frac{T_o}{T}\right) = \frac{T_o}{T} - 1 - c\left(\frac{T_o}{T}\right)^{\frac{\gamma+1}{\gamma-1}}$$

where $c = \dfrac{(T_o/T_1) - 1}{(T_o/T_1)^{\frac{\gamma+1}{\gamma-1}}} = \dfrac{\frac{\gamma - 1}{2} M_1^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma+1}{\gamma-1}}}$

In this problem $c = 0.003735$. The results of the computations are

T_o/T	f	f′	$(T_o/T)_{new}$	M_{min}
1.0000	-0.0037	0.9776	1.0038	0.138219
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
1.0038	0.0000	0.9772	1.0038	0.138235
				answer

So

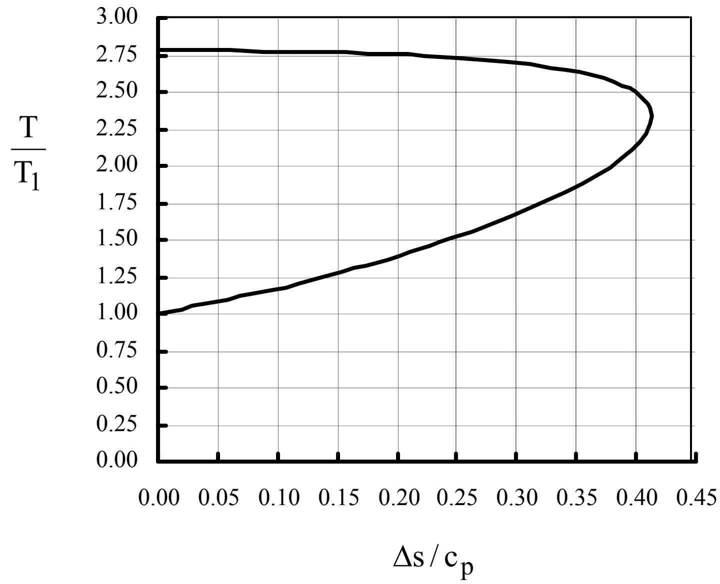
$$\left(\frac{T_o}{T}\right)_{\Delta s=0} = 1.0038 = 1 + \frac{\gamma - 1}{2} M_{\Delta s=0}^2$$

or

$$M_{\Delta s=0} = 0.1382$$

The coordinates for the Fanno-Line at this reference state are shown in the following table. The figure shown below is a plot of this data.

M	$\Delta s/c_p$	T/T_1
0.1382	0.0000	2.7893
0.33	0.2327	2.7407
0.52	0.3366	2.6564
0.71	0.3886	2.5432
0.90	0.4099	2.4086
1.09	0.4104	2.2607
1.28	0.3959	2.1066
1.47	0.3702	1.9521
1.66	0.3363	1.8017
1.86	0.2963	1.6584
2.05	0.2518	1.5240
2.24	0.2042	1.3995
2.43	0.1546	1.2852
2.62	0.1036	1.1808
2.81	0.0519	1.0860
3.00	0.0000	1.0000



Problem 2. – Draw the T-s diagram for the adiabatic flow of a gas with γ = 1.3 in a constant diameter pipe with friction. The reference Mach number, M_1, for the flow is 4.0.

Following the same procedure as indicated in Problem 1

$$\frac{T_o}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 = 1 + \left(\frac{1.3 - 1}{2}\right)(4)^2 = 2.8000$$

$$f\left(\frac{T_o}{T}\right) = \frac{T_o}{T} - 1 - c\left(\frac{T_o}{T}\right)^{\frac{\gamma+1}{\gamma-1}}$$

$$\text{where } c = \frac{(T_o/T_1) - 1}{(T_o/T_1)^{\frac{\gamma+1}{\gamma-1}}} = \frac{\frac{\gamma - 1}{2} M_1^2}{\left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma+1}{\gamma-1}}}$$

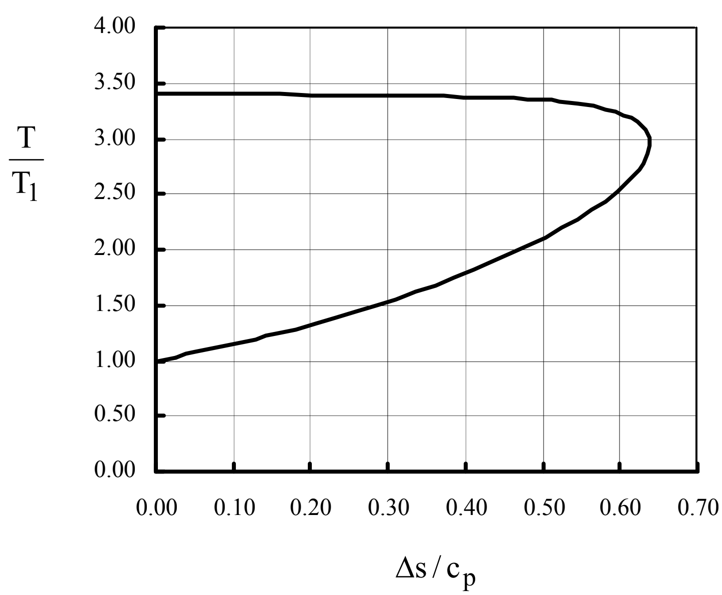
In this problem c = 0.0002021. The results of the computations are

T_o/T	f	df/dt	$(T_o/T)_{new}$	M_{min}
1.0000	-0.0002	0.9985	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367

1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
1.0002	0.0000	0.9984	1.0002	0.0367
				answer

The coordinates for the Fanno-Line at this reference state are shown in the following table. The figure shown below is a plot of this data.

M	$\Delta s/c_p$	T/T_1
0.0367	0.0000	3.3993
0.30	0.4736	3.3544
0.57	0.5896	3.2445
0.83	0.6326	3.0820
1.09	0.6373	2.8828
1.36	0.6173	2.6634
1.62	0.5800	2.4379
1.89	0.5308	2.2169
2.15	0.4732	2.0075
2.41	0.4102	1.8137
2.68	0.3437	1.6374
2.94	0.2752	1.4787
3.21	0.2059	1.3370
3.47	0.1366	1.2109
3.74	0.0678	1.0991
4.00	0.0000	1.0000



Problem 3. – Air (γ = 1.4) flows into a constant-area insulated duct with a Mach number of 0.20. For a duct diameter of 1 cm and friction coefficient of 0.02, determine the duct length required to reach Mach 0.60. Determine the length required to attain Mach 1. Finally if an additional 75 cm is added to the duct length needed to reach Mach 1, while the initial stagnation conditions are maintained, determine the reduction in flow rate that would occur.

Using the Fanno flow and isentropic flow relations we have at the upstream location

$$\begin{matrix} M_1 = 0.2 \\ \gamma = 1.4 \end{matrix} \left\{ \begin{aligned} &\left(\frac{fL_{max}}{D}\right)_1 = 14.5333 \\ &\frac{p_1}{p_{o1}} = 0.9725 \\ &\frac{T_1}{T_{o1}} = 0.9921 \end{aligned} \right.$$

and at the downstream location

$$\left.\begin{matrix} M_2 = 0.6 \\ \gamma = 1.4 \end{matrix}\right\} \left(\frac{fL_{max}}{D}\right)_2 = 0.4908$$

Thus,

$$\frac{fL}{D} = \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_2 = 14.5333 - 0.4908 = 14.0425$$

Since f = 0.02 and D = 1 cm, L = (14.0425)(1)/(0.02) = 702.1250cm = 7.0213 m

To reach Mach 1 at the exit

$$\left.\begin{matrix} M_2 = 1.0 \\ \gamma = 1.4 \end{matrix}\right\} \left(\frac{fL_{max}}{D}\right)_2 = 0$$

Thus,

$$\frac{fL}{D} = \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_2 = 14.5333 - 0.0 = 14.5333$$

Since f = 0.02 and D = 1 cm, L = (14.5333)(1)/(0.02) = 726.6650 cm.

Now if 75 cm is added to this duct length, the flow rate will be reduced (M_1 will be reduced). To determine the reduced value of M_{1R} we compute

$$\left(\frac{fL_{max}}{D}\right)_{1R} = \frac{0.02(726.6650+75)}{1} = \frac{0.02(801.6650)}{1} = 16.0333$$

From this value we find that M_{1R} = 0.1917. Note the subscript R has been added to indicate the reduced value. Using the isentropic flow relations we have

$$\left.\begin{array}{l} M_{1R} = 0.1917 \\ \gamma = 1.4 \end{array}\right\} \begin{array}{l} \left(\frac{fL_{max}}{D}\right)_{1R} = 16.0333 \\ \frac{p_{1R}}{p_{o1}} = 0.9747 \\ \frac{T_{1R}}{T_{o1}} = 0.9927 \end{array}$$

The original mass flow rate and the reduced flow rate may be written respectively as

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1}$$

$$\dot{m}_R = \rho_{1R} A V_{1R} = \left(\frac{p_{1R}}{RT_{1R}}\right) A M_{1R} \sqrt{\gamma R T_{1R}}$$

Since the stagnation conditions are maintained we may write the following

$$\frac{\dot{m}_R}{\dot{m}} = \frac{\left(\frac{p_{1R}}{p_{o1}}\right)\left(\frac{T_1}{T_{o1}}\right)}{\left(\frac{p_1}{p_{o1}}\right)\left(\frac{T_{1R}}{T_{o1}}\right)}\left(\frac{M_{1R}}{M_1}\right)\sqrt{\frac{\left(\frac{T_{1R}}{T_{o1}}\right)}{\left(\frac{T_1}{T_{o1}}\right)}} = \frac{\left(\frac{p_{1R}}{p_{o1}}\right)}{\left(\frac{p_1}{p_{o1}}\right)}\left(\frac{M_{1R}}{M_1}\right)\sqrt{\frac{\left(\frac{T_1}{T_{o1}}\right)}{\left(\frac{T_{1R}}{T_{o1}}\right)}}$$

$$= \frac{(0.9747)}{(0.9725)}\frac{0.1917}{0.2}\sqrt{\frac{0.9921}{0.9927}} = 0.9604$$

So the % reduction is (1 – 0.9604)100 = 3.9622%

Problem 4. – Air (γ = 1.4 and R = 0.287 kJ/kg · K) enters a constant-area insulated duct with a Mach number of 0.35, a stagnation pressure of 105 kPa, and stagnation temperature of 300 K. For a duct length of 50 cm, duct diameter of 1 cm, and friction coefficient of 0.022, determine the air force on the duct wall.

A force-momentum balance on a control volume within the duct reveals that

$$p_1A_1 - p_2A_2 - F_{\text{wall on air}} = \dot{m}(V_2 - V_1)$$

Thus to compute the force we must first determine the entry and exit values of the static pressure and velocity as well as the mass flow rate.

Using the Fanno flow and isentropic flow relations we have at the upstream location

$$\begin{matrix} M_1 = 0.35 \\ \gamma = 1.4 \end{matrix} \left\{ \begin{aligned} &\left(\frac{fL_{max}}{D}\right)_1 = 3.4525 \\ &\frac{p_1}{p^*} = 3.0922 \\ &\frac{\rho_1}{\rho^*} = \frac{V^*}{V_1} = 2.6400 \\ &\frac{p_1}{p_{o1}} = 0.9188 \\ &\frac{T_1}{T_{o1}} = 0.9761 \end{aligned} \right.$$

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 3.4525 - \frac{(0.022)50}{1} = 2.3525$$

From this value we find that $M_2 = 0.3976$. Using the Fanno and isentropic flow relations we have

$$\begin{matrix} M_2 = 0.3976 \\ \gamma = 1.4 \end{matrix} \left\{ \begin{aligned} &\frac{p_2}{p^*} = 2.7126 \\ &\frac{\rho_2}{\rho^*} = \frac{V^*}{V_2} = 2.3320 \\ &\frac{T_2}{T_{o2}} = 0.9694 \end{aligned} \right.$$

Since $p_{o1} = 105$ kPa,

$$p_1 = \frac{p_1}{p_{o1}} p_{o1} = (0.9188)(105) = 96.4740 \text{ kPa}$$

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1} p_1 = \frac{2.7126}{3.0922} 96.4740 = 84.6308 \text{ kPa}$$

Because the flow is adiabatic: $T_{o1} = T_{o2} = 300$ K.

$$T_1 = \frac{T_1}{T_{o1}} T_{o1} = (0.9761)(300) = 292.8300 \text{ K}$$

$$V_1 = M_1 a_1 = 0.35\sqrt{1.4(287)(292.8300)} = 120.0551 \,\text{m/s}$$

$$V_2 = \left(\frac{V_2}{V^*}\right)\left(\frac{V^*}{V_1}\right)V_1 = \left(\frac{\rho^*}{\rho_2}\right)\left(\frac{\rho_1}{\rho^*}\right)V_1 = \left(\frac{2.6400}{2.3320}\right)120.0551 = 135.9114 \,\text{m/s}$$

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1} = \left[\frac{(96.4740)}{(0.287)(292.83)}\right]\left(\frac{\pi}{4}10^{-4}\right)(120.0551)$$
$$= 0.01082 \,\text{kg/s}$$

Finally,

$$\begin{aligned} F_{\text{wall on air}} &= p_1 A_1 - p_2 A_2 - \dot{m}(V_2 - V_1) \\ &= (96.4740 - 84.6308)\left(\frac{\pi}{4}10^{-4}\right)10^3 \frac{\text{N}}{\text{kN}} - (0.01082)(135.9114 - 120.0551) \\ &= 0.9302 - 0.1716 = 0.7585 \,\text{N} \end{aligned}$$

Problem 5. – Hydrogen (γ = 1.4 and R = 4124 J/kg · K) enters a constant-area insulated duct with a velocity of 2600 m/s, static temperature of 300 K, and stagnation pressure of 520 kPa. The duct is 2 cm in diameter, and 10 cm long. For a friction coefficient of 0.02, determine the change of static pressure and temperature in the duct and the exit velocity of the hydrogen.

$$M_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt{\gamma R T_1}} = \frac{2600}{\sqrt{1.4(4124)300}} = 1.9756$$

Using the Fanno flow and isentropic relations we have at the upstream location

$$M_1 = 1.9756 \quad \gamma = 1.4 \quad \left\{ \begin{array}{l} \left(\dfrac{fL_{max}}{D}\right)_1 = 0.2977 \\ \dfrac{p_1}{p^*} = 0.4155 \\ \dfrac{\rho_1}{\rho^*} = \dfrac{V^*}{V_1} = 0.6166 \\ \dfrac{T_1}{T^*} = 0.6739 \\ \dfrac{p_1}{p_{o1}} = 0.1327 \end{array} \right.$$

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 0.2977 - \frac{(0.02)10}{2} = 0.1977$$

From this value we find that M_2 = 1.6712. Using the Fanno and isentropic flow relations we have

$$M_2 = 1.6712 \quad \gamma = 1.4 \quad \left\{ \begin{array}{l} \dfrac{p_2}{p^*} = 0.5251 \\ \dfrac{\rho_2}{\rho^*} = \dfrac{V^*}{V_2} = 0.6820 \\ \dfrac{T_2}{T^*} = 0.7699 \end{array} \right.$$

$$\frac{p_2}{p_1} = \frac{p_2}{p^*}\frac{p^*}{p_1} = \frac{0.5251}{0.4155} = 1.2638$$

$$\frac{T_2}{T_1} = \frac{T_2}{T^*}\frac{T^*}{T_1} = \frac{0.7699}{0.6739} = 1.1425$$

$$V_2 = \left(\frac{V_2}{V^*}\right)\left(\frac{V^*}{V_1}\right)V_1 = \left(\frac{\rho^*}{\rho_2}\right)\left(\frac{\rho_1}{\rho^*}\right)V_1 = \left(\frac{0.6166}{0.6820}\right)2600 = (0.9041)2600$$
$$= 2350.6745\,\text{m/s}$$

$$p_2 - p_1 = p_1\left(\frac{p_2}{p_1} - 1\right) = \frac{p_1}{p_{o1}}p_{o1}\left(\frac{p_2}{p_1} - 1\right) = (0.1327)(520)(1.2638 - 1)$$
$$= 18.2033\,\text{kPa}$$

$$T_2 - T_1 = T_1\left(\frac{T_2}{T_1} - 1\right) = 300(1.1425 - 1) = 42.7500\,\text{K}$$

Problem 6. – A constant-area duct, 25 cm in length by 1.3 cm in diameter, is connected to an air reservoir through a converging nozzle, as shown in Figure P9.6. For a constant reservoir pressure of 1 MPa and constant reservoir temperature of 600 K, determine the flow rate through the duct for a back pressure of 101 kPa. Assume adiabatic flow in the tube with f = 0.023.

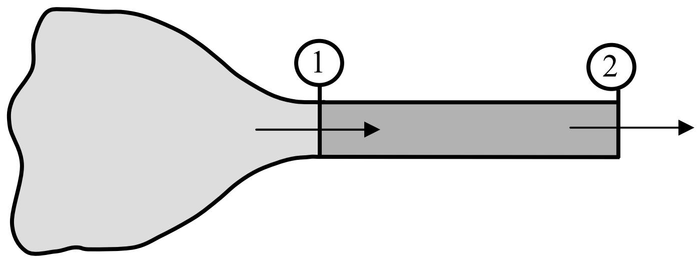


Figure P9.6

First determine the exit pressure assuming the duct is choked. Therefore,

$$\frac{fL}{D} = \frac{(0.023)(25)}{(1.3)} = 0.4423 = \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - 0.0 = \left(\frac{fL_{max}}{D}\right)_1$$

From this value we can determine that $M_1 = 0.6129$. At this Mach number using the isentropic and Fanno flow pressure relations we may write that

$$p_e = p_2 = p^* = \left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{o1} = \left(\frac{1}{1.7239}\right)(0.7761)1000 = 450.2001\,\text{kPa}$$

Since the back pressure is well below this value the assumption that the duct is choked is correct and we may proceed to determine the flow rate. Now at $M_1 = 0.6129$,

$$T_1 = \left(\frac{T_1}{T_{o1}}\right)T_{o1} = (0.9301)600 = 558.0600\,\text{K}$$

$$p_1 = \left(\frac{p_1}{p_{o1}}\right)p_{o1} = (0.7761)1000 = 776.1\,\text{kPa}$$

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1}$$

$$= \left[\frac{(776.1)}{(0.287)(558.06)}\right]\left(\frac{\pi}{4} 0.013^2\right)(0.6129)\sqrt{(1.4)(287)(558.06)}$$

$$= 0.1867 \text{ kg/s}$$

***Problem** 7.* – Find the time required for the pressure in the tank filled with Nitrogen (γ = 1.4 and R = 296.8 J/kg · K) shown in Figure P9.7 to drop from 1 MPa to 500 kPa. The tank volume is 8 m^3 and the tank temperature is 300K. Assume the tank temperature remains constant and the flow in the 3 m long, 1 cm diameter connecting tube is adiabatic with f = 0.018. The back pressure is 101 kPa.

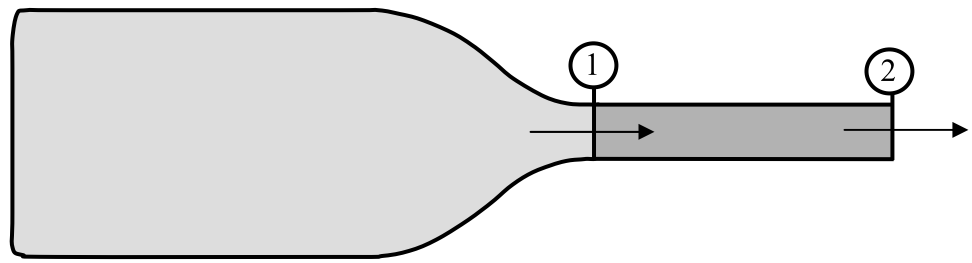


Figure P9.7

First determine the exit pressure assuming the duct is choked. Therefore,

$$\frac{fL}{D} = \frac{(0.018)(3)}{(0.01)} = 5.4000 = \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - 0.0 = \left(\frac{fL_{max}}{D}\right)_1$$

From this value we can determine that M_1 = 0.2979. At this Mach number using the isentropic and Fanno flow pressure relations we may write that

$$p_2 = p^* = \left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)\left(\frac{p_{o1}}{p_r}\right) p_r = \left(\frac{1}{3.6451}\right)(0.9403)(1) p_r(t) = 0.2580 p_r(t) \text{kPa}$$

The lowest value of reservoir pressure is 500 kPa; therefore, the smallest value of p* is (0.2580)500 = 129 kPa and since this value is above p_b = 101 kPa, the duct is choked for the entire process.

At M_1 = 0.2979,

$$T_1 = \left(\frac{T_1}{T_{o1}}\right) T_{o1} = (0.9826)300 = 294.7800\,\text{K}$$

$$p_1 = \left(\frac{p_1}{p_{o1}}\right) p_{o1} = (0.9403) p_r$$

Thus,

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1}$$

$$= \left[\frac{(0.9403) p_r}{(0.2968)(294.78)}\right]\left(\frac{\pi}{4} 0.01^2\right)(0.2979)\sqrt{(1.4)(296.8)(294.78)}$$

$$= 8.8006\text{x}10^{-5}\, p_r$$

Now within the reservoir,

$$p_r \forall_r = m_r RT$$

Taking the time derivative of this expression gives

$$\frac{dp_r}{dt} = \left(\frac{RT}{\forall_r}\right)\frac{dm_r}{dt} = \frac{(0.2968)300}{8}\frac{dm_r}{dt} = 11.1300\frac{dm_r}{dt}$$

From a mass balance on the reservoir,

$$\frac{dm_r}{dt} = -\dot{m}$$

Therefore,

$$\frac{dp_r}{dt} = -(11.1300)\left(8.8006\text{x}10^{-5}\right) p_r = -9.7950\text{x}10^{-4}\, p_r$$

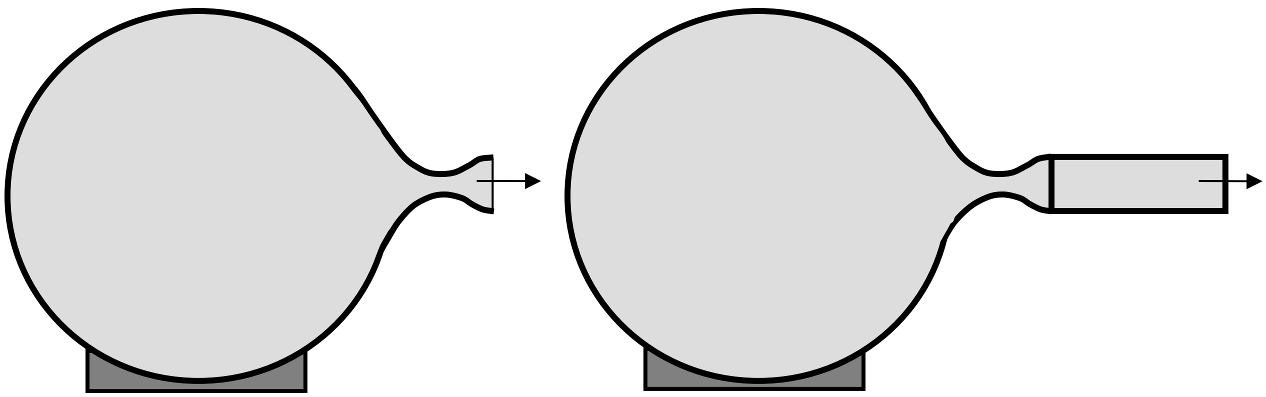
Integration gives

$$\ln\frac{(p_r)_f}{(p_r)_i} = \ln\frac{500}{1000} = -0.6931 = -9.7950\text{x}10^{-4}\, t$$

Hence,

$$t = \left(\frac{0.6931}{9.7950}\right)10^4 = 707.6052\,\text{s}$$

Problem 8. – A converging-diverging nozzle has an area ratio of 3.3, i.e., the exit and therefore the duct area is 3.3 times the throat area, which is 60 cm^2. The nozzle is supplied from a tank containing air (γ = 1.4 and R = 0.287 kJ/kg · K) at 100 kPa at 270 K. For case A of Figure P9.8, find the maximum mass flow possible through the nozzle and the range of back pressures over which the mass flow can be attained. Repeat for case B, in which a constant-area insulated duct of length 1.5 m and f = 0.022 is added to the nozzle.



Case A **Case B**

Figure P9.8

Case A

The maximum flow rate will occur when the throat Mach number is 1. At this Mach number, the throat static to total pressure and temperature ratios are: 0.5283 and 0.8333, respectively. Accordingly, the flow rate is computed to be

$$\dot{m}_{max} = \rho_t A_t V_t = \left(\frac{p_t}{RT_t}\right) AM_t \sqrt{\gamma RT_t} = \left(\frac{\frac{p_t}{p_o} p_o}{R\frac{T_t}{T_o} T_o}\right) AM_t \sqrt{\gamma R\left(\frac{T_t}{T_o}\right) T_o}$$

$$= \left[\frac{(0.5283)(100)}{(0.287)(0.8333)(270)}\right]\left(60\text{x}10^{-4}\right)(1)\sqrt{(1.4)(287)(0.8333)270}$$

$$= 1.4760\,\text{kg/s}$$

For A/A* = 3.3, we can determine that the exit Mach number is 0.1787. At this value the exit static to total pressure ratio is 0.9780. Thus, the maximum flow rate will occur for

$$0 \le p_b \le (0.9780)100 = 97.7952\,\text{kPa}$$

Case B

Here too, the maximum flow rate will occur when the throat Mach number is 1. At this Mach number, the throat static to total pressure and temperature ratios are: 0.5283 and 0.8333, respectively. Accordingly, the maximum flow rate will be the same as that in Case A, viz., 1.4760 kg/s.

Now for subsonic flow at the nozzle exit, and the duct inlet, $M_1 = 0.1787$.

$$\left(\frac{fL_{max}}{D}\right)_1 = 18.8522$$

The diameter of the duct is computed as follows

$$D = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{4(60)(3.3)}{\pi}} = 15.8777\,\text{cm}$$

Thus

$$\frac{fL}{D} = \frac{(0.022)150}{15.8777} = 0.2078$$

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 18.8522 - 0.2078 = 18.6444$$

So the exit Mach number is 0.1796. The exit pressure which is equal to the back pressure is computed as follows

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}\frac{p_1}{p_{o1}}p_o = (6.0808)\left(\frac{1}{6.1106}\right)(0.977952)100 = 97.3183\,\text{kPa}$$

Thus, the maximum flow rate will occur for

$$0 \le p_b \le 97.3183\,\text{kPa}$$

Problem 9. – A 3-m^3 volume tank, R, is to be filled to a pressure of 200 kPa (initial pressure 0 kPa). The tank is connected to a reservoir tank, L, containing air at 3 MPa and 300 K, whose volume is also 3 m^3. A 30-m length of 2.5 cm-diameter tubing is used to connect the two vessels, as shown in Figure P9.9. Determine the time required to fill the tank to 200 kPa. Assume Fanno flow with f = 0.02.

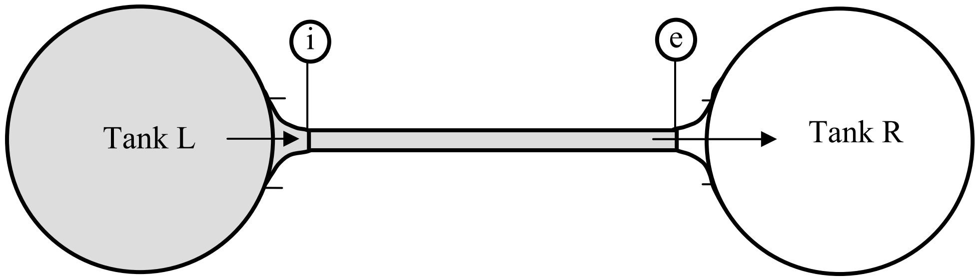


Figure P9.9

Because R, the tank on the right, is evacuated, it may be safely assumed that $M_e = 1$. Therefore,

$$\frac{fL}{D} = \frac{(0.02)(30)}{(0.025)} = 24.0 = \left(\frac{fL_{max}}{D}\right)_i - \left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_i - 0.0 = \left(\frac{fL_{max}}{D}\right)_i$$

From this value we find that $M_i = 0.1606$. Using the isentropic flow relations we have

$$\begin{matrix} M_i = 0.1606 \\ \gamma = 1.4 \end{matrix} \left\{ \begin{aligned} \left(\frac{p}{p_o}\right)_i &= 0.98215 \\ \left(\frac{T}{T_o}\right)_i &= 0.9949 \end{aligned} \right.$$

Thus,

$$\begin{aligned} \dot{m} &= \rho_i A_i V_i = \left(\frac{p_i}{RT_i}\right) A_i M_i \sqrt{\gamma R T_i} \\ &= \left[\frac{(0.98215)p_{oR}}{(0.287)(0.9949)(300)}\right]\left(\frac{\pi}{4}0.025^2\right)(0.1606)\sqrt{(1.4)(287)(0.9949)(300)} \\ &= 3.1302\text{x}10^{-4}\, p_{oR} \end{aligned}$$

Now in order that there be Fanno flow, T_o must remain constant. So for tank L

$$p_{oL} \forall = m_L R T_o$$

Differentiate this with respect to time to get

$$\forall \frac{dp_{oL}}{dt} = RT_o \frac{dm_L}{dt}$$

So

$$\frac{dm_L}{dt} = \left(\frac{\forall}{RT_o}\right)\frac{dp_{oL}}{dt} = -\dot{m}_i = -3.1302\text{x}10^{-4} p_{oL} = \frac{3}{(0.287)(300)}\frac{dp_{oL}}{dt} = 0.03484 \frac{dp_{oL}}{dt}$$

Similarly, for the tank on the right

$$p_{oR}\forall = m_R RT_o$$

Therefore,

$$\frac{dm_R}{dt} = \left(\frac{\forall}{RT_o}\right)\frac{dp_{oR}}{dt} = \dot{m}_e = \dot{m}_i = -\left(\frac{\forall}{RT_o}\right)\frac{dp_{oL}}{dt}$$

Clearly,

$$\frac{dp_{oR}}{dt} = -\frac{dp_{oL}}{dt}$$

That is

$$p_{oR}\Big|_0^{200} = 200\text{kPa} = p_{oL2} - p_{oL1}$$

$$\frac{dp_{oL}}{dt} = -\frac{3.1302\text{x}10^{-4}}{0.03484}p_{oL} = -8.9845\text{x}10^{-3}p_{oL}$$

Integration brings

$$\Delta t = -\int_{3\text{MPa}}^{2.8\text{MPa}}\frac{10^3}{8.9845}\frac{dp_{oL}}{p_{oL}} = 111.3028\ln\left(\frac{3}{2.8}\right) = 7.6791\text{s}$$

Problem 10. Find the mass flow rate of air (γ = 1.4 and R = 0.287 kJ/kg · K) through the system shown in Figure P9.10. Assume Fanno line flow in the duct and isentropic flow in the converging sections; f = 0.01.

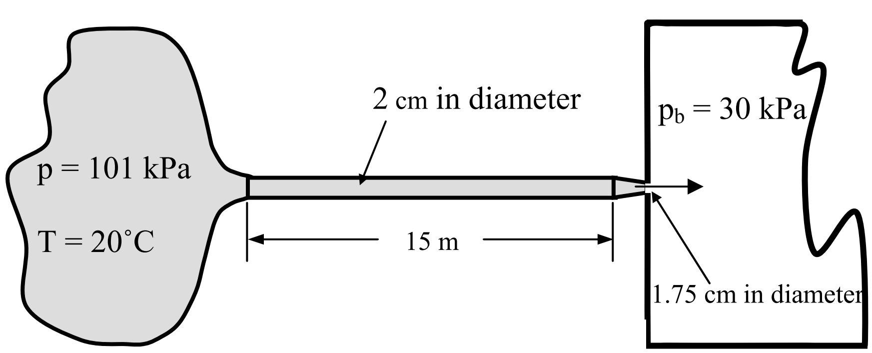
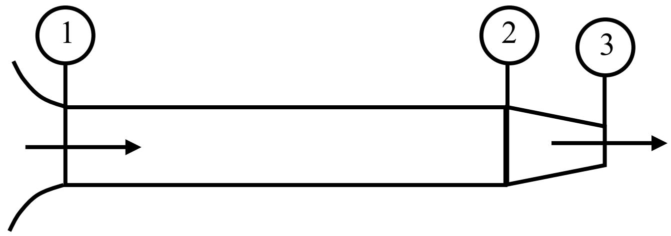


Figure P9.10



Assume that the system is choked so that $M_3 = 1$, $A_3 = A^*$. So $(A/A^*) = (A_2/A_3) = (2/1.75)^2 = 1.3061$, which yields $M_2 = 0.5184$. In turn, $(fL_{max}/D)_2 = 0.9287$. Also, $(fL/D) = (0.01)(15)/(0.02) = 7.5$; therefore,

$$\left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL_{max}}{D}\right)_2 + \frac{fL}{D} = 0.9287 + 7.5 = 8.4287$$

From this, we find that $M_1 = 0.2506$. Consequently, with M_1, M_2 and M_3 we are able to compute the following static pressures

$$p_1 = \left(\frac{p_1}{p_{o1}}\right)p_{o1} = (0.95725)101 = 96.6823\,\text{kPa}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 = (2.0585)\left(\frac{1}{4.3441}\right)96.6823 = 45.8140\,\text{kPa}$$

$$p_3 = \left(\frac{p_3}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_2}\right)p_2 = (0.5283)(1.0)\left(\frac{1}{0.83257}\right)45.8140 = 29.0708\,\text{kPa}$$

Since $p_3 = p_e$ and since p_e must be equal to or greater than p_b, the nozzle is not choked as assumed. Therefore, $p_3 = p_e = 30$ kPa and $M_3 = M_e < 1$.

Assume $M_3 = 0.9$ and from the isentropic relations we find

$$\frac{A_3}{A^*} = 1.0089, \quad \frac{p_3}{p_{o3}} = 0.59126$$

Now

$$\frac{A_2}{A^*} = \frac{A_2}{A_3}\frac{A_3}{A^*} = \left(\frac{2.0}{1.75}\right)^2 1.0089 = 1.3177$$

From this we find $M_2 = 0.5119$, from which we obtain

$$\left(\frac{fL_{max}}{D}\right)_2 = 0.9761, \quad \frac{p_2}{p_{o2}} = 0.8363, \quad \frac{p_2}{p^*} = 2.0860$$

$$\left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL_{max}}{D}\right)_2 + \frac{fL}{D} = 0.9761 + 7.5 = 8.4761$$

From this, we find that $M_1 = 0.2501$. Consequently, with M_1, M_2 and M_3 we are able to compute the following static pressures

$$p_1 = \left(\frac{p_1}{p_{o1}}\right)p_{o1} = (0.9574)101 = 96.6974\,\text{kPa}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 = (2.0860)\left(\frac{1}{4.3531}\right)96.6974 = 46.3373\,\text{kPa}$$

$$p_3 = \left(\frac{p_3}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_2}\right)p_2 = (0.59126)(1.0)\left(\frac{1}{0.8363}\right)46.6974 = 32.7602\,\text{kPa}$$

Too large; therefore M_3 needs to be increased. After a few tries $M_3 = 0.973$ and from the isentropic relations we find

$$\frac{A_3}{A^*} = 1.0006, \quad \frac{p_3}{p_{o3}} = 0.5450$$

Now

$$\frac{A_2}{A^*} = \frac{A_2}{A_3}\frac{A_3}{A^*} = \left(\frac{2.0}{1.75}\right)^2 1.0006 = 1.3069$$

From this we find $M_2 = 0.5180$, from which we obtain

$$\left(\frac{fL_{max}}{D}\right)_2 = 0.9316, \quad \frac{p_2}{p_{o2}} = 0.8328, \quad \frac{p_2}{p^*} = 2.0602$$

$$\left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL_{max}}{D}\right)_2 + \frac{fL}{D} = 0.9316 + 7.5 = 8.4316$$

From this, we find that $M_1 = 0.2506$. Consequently, with M_1, M_2 and M_3 we are able to compute the following static pressures

$$p_1 = \left(\frac{p_1}{p_{o1}}\right)p_{o1} = (0.9572)101 = 96.6772\,\text{kPa}$$

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)p_1 = (2.0602)\left(\frac{1}{4.3438}\right)96.6772 = 45.8521\,\text{kPa}$$

$$p_3 = \left(\frac{p_3}{p_{o3}}\right)\left(\frac{p_{o3}}{p_{o2}}\right)\left(\frac{p_{o2}}{p_2}\right)p_2 = (0.5450)(1.0)\left(\frac{1}{0.8328}\right)45.8521 = 30.0082\,\text{kPa}$$

Slightly too high but close enough. Note from the isentropic relations at M_1, $(T/T_o)_1$ = (0.9876)293 = 289.3668K

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right)AM_1\sqrt{\gamma RT_1}$$

$$= \left[\frac{(96.6772)}{(0.287)289.3668}\right]\left(\frac{\pi}{4}0.02^2\right)0.2506\sqrt{(1.4)(287)(289.3668)}$$

$$= 0.0313\,\text{kg/s}$$

Problem 11. – For the flow of air ($\gamma = 1.4$ and R = 0.287 kJ/kg · K) from the reservoir at 650 kPa and 1000 K shown in Figure P9.11, assume isentropic flow in the convergent-divergent nozzle and Fanno flow in the constant-area duct, which has a length of 20 cm and a diameter of 1 cm. The area ratio A_2/A_1 of the C-D nozzle is 2.9. Take the friction factor to be 0.02.

(a) Find the mass flow rate for a back pressure of 0 kPa.
(b) For part (a), find the pressure at the exit plane of the duct.
(c) Find the back pressure necessary for a normal shock to occur at the exit plane of the nozzle (2).
(d) Find the back pressure necessary for a normal shock to appear just downstream of the nozzle throat (1).

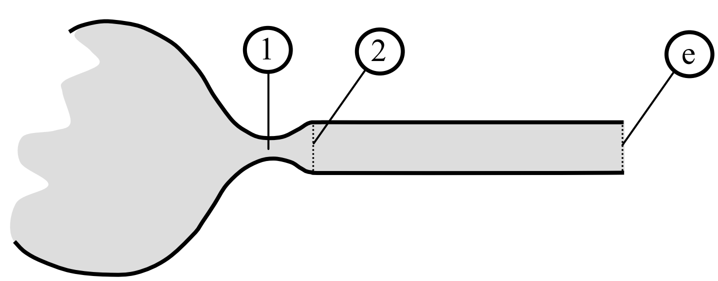


Figure P9.11

(a) For $A_2/A_1 = A_2/A^* = 2.9$, $M_2 = 2.6015$. Therefore, $(fL_{max}/D)_2 = 0.45288$. Now $fL/D = (0.02)(20)/(1) = 0.4$. Hence, $L < L_{max}$ so the flow cannot reach $M_e = 1$. To compute the exit Mach number we have

$$\left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_2 - \frac{fL}{D} = 0.45288 - 0.4 = 0.05288$$

From which we find $M_e = 1.2636$. Now at the nozzle throat, M = 1 so

$$p_t = \left(\frac{p_t}{p_{ot}}\right)p_{ot} = (0.5283)650 = 343.3950\,\text{kPa}$$

$$T_t = \left(\frac{T_t}{T_{ot}}\right)T_{ot} = (0.8333)1000 = 833.3333\,\text{kPa}$$

$$A_t = \frac{\frac{\pi}{4}(0.01)^2}{2.9} = 2.7083\text{x}10^{-5}\,\text{m}^2$$

$$\dot{m} = \rho_t A_t V_t = \left(\frac{p_t}{RT_t}\right)A_t M_t \sqrt{\gamma R T_t}$$

$$= \left[\frac{343.3950}{(0.287)(833.3333)}\right]\left(2.7083\text{x}10^{-5}\right)(1)\sqrt{(1.4)(287)(833.3333)}$$

$$= 0.02250\,\text{kg/s}$$

(b)

$$p_2 = \left(\frac{p_2}{p_{o2}}\right)p_{o2} = (0.050001)650 = 32.5007\,\text{kPa}$$

$$p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_2}\right)p_2 = (0.7548)\left(\frac{1}{0.27448}\right)32.5007 = 89.3744\,\text{kPa}$$

(c) In this case a shock stands at the nozzle exit – station 2. We will call the duct inlet, (on the downstream side of the shock), station 3. Now at $M_2 = 2.60147$, from the normal shock relations $M_3 = 0.50374$ and $p_3/p_2 = 7.7289$. Therefore,

$$p_3 = \left(\frac{p_3}{p_2}\right)p_2 = (7.7289)32.5007 = 251.1947\,\text{kPa}$$

Also from the Fanno relations at M_3, $(fL_{max}/D)_3$ = 1.03895. Since fL/D = 0.4 it follows that

$$\left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_3 - \frac{fL}{D} = 1.03895 - 0.4 = 0.63895$$

From which we find $M_e = 0.5668$.

$$p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_3}\right)p_3 = (1.87361)\left(\frac{1}{2.12146}\right)251.1947 = 221.8476\,\text{kPa}$$

(d) In this case a shock appears just downstream of the nozzle throat. Consequently, subsonic flow exits the nozzle. For $A_2/A_1 = A_2/A^* = 2.9$, $M_2 = 0.2046$. Therefore, $(fL_{max}/D)_2 = 13.7780$. Since, fL/D = 0.4, then $(fL_{max}/D)_e = 13.7780 - 0.4 = 13.3780$ from which we find $M_e = 0.2072$

$$p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_2}\right)\left(\frac{p_2}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{ot}}\right)p_{ot} = (5.26503)\left(\frac{1}{5.33181}\right)(0.97124)650 = 623.3990\,\text{kPa}$$

Problem 12. – In which configuration of Figure P9.12, (a) or (b), will the high-pressure tank empty faster? Explain.

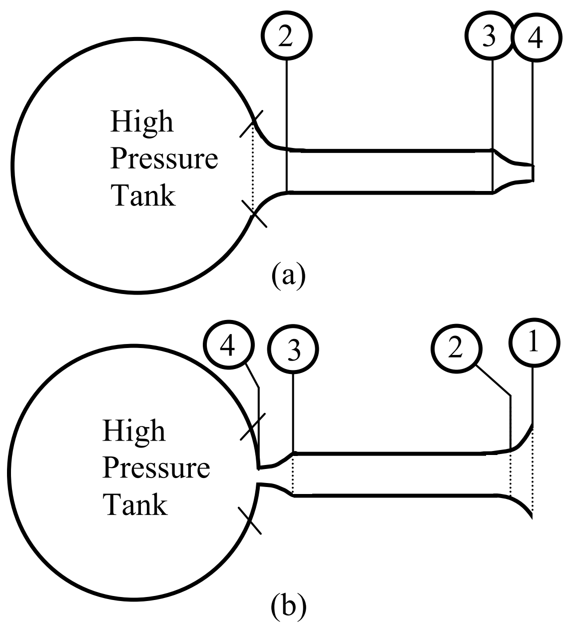


Figure P9.12

Both (a) and (b) are choked at section 4, thus, $M_4 = 1$. However, because of the loss of total pressure, p_{o4} is smaller in (a) than in (b). This results in a smaller $\dot{m}$ for the tank in (a); hence, tank (b) will empty faster than (a).

Problem 13. – Air (γ = 1.4 and R = 0.287 kJ/kg · K) flows through a converging-diverging nozzle with area ratio of 2.9 (Figure P9.13), which exhausts into a constant-area insulated duct with a length of 50 cm and diameter of 1 cm. If the system back pressure is 50 kPa, determine the range of reservoir pressures over which a normal shock will appear in the duct. Let f = 0.02 in the duct.

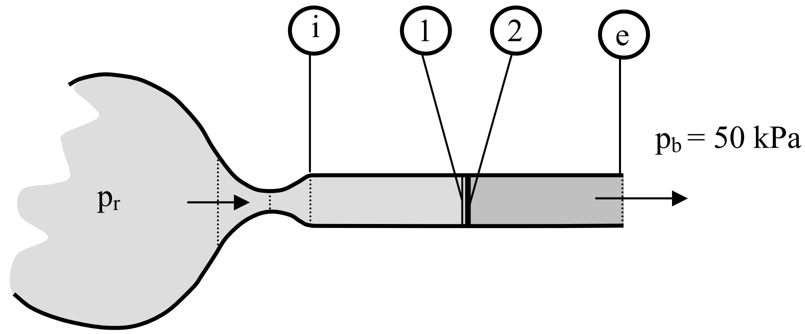


Figure P9.13

Now fL/D = (0.02)(50)/(1) = 1.0

Shock at Duct Inlet

For $A_1/A_t = A_1/A^* = 2.9$, $M_1 = 2.6015$. From the isentropic relations at this Mach number we obtain $p_1/p_{o1} = 0.0500$. From the normal shock relations at this Mach number we obtain $M_2 = 0.50374$ and $p_2/p_1 = 7.7289$. From Fanno flow relations at M_2 we obtain $(fL_{max}/D)_2 = 1.03895$. Therefore,

$$\left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_2 - \frac{fL}{D} = 1.03895 - 1.0 = 0.03895$$

From which we find $M_e = 0.8455$. Consequently,

$$p_2 = \left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_e}\right)p_e = (2.12146)\left(\frac{1}{1.21194}\right)50 = 87.5233\,\text{kPa}$$

$$p_r = p_{o1} = \left(\frac{p_{o1}}{p_1}\right)\left(\frac{p_1}{p_2}\right)p_2 = \left(\frac{1}{0.05}\right)\left(\frac{1}{7.7289}\right)(87.5233) = 226.4832\,\text{kPa}$$

Shock at Duct Exit

Since, at M_1 = 2.6015, $(fL_{max}/D)_1$ = 0.4529, we see that L > L_{max} , hence a shock will exist within the duct for reservoir pressures that exceed 226.4832 kPa. Using the method described in Example 9.5 we can determine that the shock will penetrate only 0.8097 cm into the duct.

Problem 14. – A converging-diverging nozzle with area ratio of 3.2 (Figure P9.14) exhausts air (γ = 1.4 and R = 0.287 kJ/kg · K) into a constant-area insulated duct with a length of 50 cm and diameter of 1 cm. If the reservoir pressure is 500 kPa, determine the range of back pressures over which a normal shock will appear in the duct (f = 0.02).

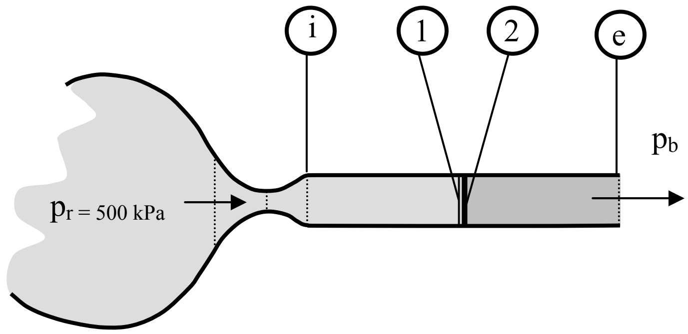


Figure P9.14

Now fL/D = (0.02)(50)/(1) = 1.0

Shock at Duct Inlet

For $A_1/A_t = A_1/A^*$ = 3.2, M_1 = 2.7056. From the isentropic relations at this Mach number we obtain p_1/p_{o1} = 0.04258. From the normal shock relations at this Mach number we obtain M_2 = 0.4952 and p_2/p_1 = 8.3737. From Fanno flow relations at M_2 we obtain $(fL_{max}/D)_2$ = 1.1090. Therefore,

$$\left(\frac{fL_{max}}{D}\right)_e = \left(\frac{fL_{max}}{D}\right)_2 - \frac{fL}{D} = 1.1090 - 1.0 = 0.1090$$

From which we find M_e = 0.7645. Consequently,

$$p_1 = \left(\frac{p_1}{p_{o1}}\right)p_{o1} = (0.04258)500 = 21.2900\,\text{kPa}$$

$$p_2 = \left(\frac{p_2}{p_1}\right)p_1 = (8.3737)21.2900 = 178.2761\ \text{kPa}$$

$$p_e = \left(\frac{p_e}{p^*}\right)\left(\frac{p^*}{p_2}\right)p_2 = (1.3559)\left(\frac{1}{2.1598}\right)(178.2761) = 111.9199\,\text{kPa}$$

Shock at Duct Exit

Since, at M_1 = 2.7056, $(fL_{max}/D)_1$ = 0.4729, we see that L > L_{max} , hence a shock will exist within the duct for back pressures that are below 111.9199 kPa. Using the method described in Example 9.5 we can determine that the shock will penetrate only 2.2360 cm into the duct.

Problem 15. – A converging-diverging nozzle is connected to a reservoir containing gas (γ = 1.4). The area ratio of the nozzle is such that the Mach number is 3.5 exiting the nozzle and entering a constant-area duct of length-to-diameter ratio, L/D, of 100 to 1 and friction coefficient of 0.01. (a) Determine the normal shock location, if the Mach number at the exit is 0.75. (b) With the shock at this location, how much longer can the duct be made before choking occurs at the exit with no change of M_i? Refer to Figure 9.19 for the nomenclature.

(a) Determining shock location:

At M_i = 3.5, $(fL_{max}/D)_i$ = 0.5864, and at M_e = 0.75, $(fL_{max}/D)_e$ = 0.1273. For the duct under consideration, fL/D = 0.01(100)/1 = 1.0; hence, L > $(L_{max})_i$. To determine the location of the shock for this case, first calculate the value of $F(M_1)$ from Eq.(9.29)

$$F(M_1) = \left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL}{D}\right) + \left(\frac{fL_{max}}{D}\right)_e - \left(\frac{fL_{max}}{D}\right)_i$$

$$= 1.0 + 0.1273 - 0.5864 = 0.5409$$

The value of M_1 can be obtained by numerically solving Eq.(9.33) using a spreadsheet program that implements the Newton-Raphson method. The following table contains the history of the iteration process:

Iteration	M	**f(M)**	**f(M+ΔM)**	**f(M-ΔM)**	**Δf/ΔM**	**M_{new}**
1	2.0000	-0.25799	0.2878	0.2779	0.4960	2.5201
2	2.5201	0.005189	0.5510	0.5411	0.4959	2.5097
3	2.5097	-5.3E-06	0.5458	0.5359	0.4969	2.5097
4	2.5097	3.76E-11	0.5458	0.5359	0.4969	2.5097
5	2.5097	0	0.5458	0.5359	0.4969	2.5097
6	2.5097	0	0.5458	0.5359	0.4969	2.5097
7	2.5097	0	0.5458	0.5359	0.4969	2.5097
8	2.5097	0	0.5458	0.5359	0.4969	2.5097
						Answer

At M_1 = 2.5097, M_2 is found from Eq.(9.31) to be 0.5121. At these Mach numbers, $(fL_{max}/D)_1$ = 0.4340 and $(fL_{max}/D)_2$ = 0.9749. The shock location is determined from Eq.(9.34)

$$\frac{L_s}{D} = \left(\frac{1}{f}\right)\left[\left(\frac{fL_{max}}{D}\right)_i - \left(\frac{fL_{max}}{D}\right)_1\right] = \frac{1}{0.01}(0.5864 - 0.4340) = 15.2397$$

(b) Determining the duct length to accelerate the flow to Mach 1 for the same shock location determined above.

Because M_e = 1.0, $(fL_{max}/D)_e$ = 0.0. Also, because the shock location is fixed $F(M_1)$ = 0.54085 and for the same inlet Mach number, i.e, M_i = 3.5, $(fL_{max}/D)_i$ = 0.5864; hence,

$$F(M_1) = 0.54085 = \left(\frac{fL}{D}\right) + \left(\frac{fL_{max}}{D}\right)_e - \left(\frac{fL_{max}}{D}\right)_i$$

$$= \frac{(0.01)L}{D} + 0.0 - 0.5864$$

Therefore, L/D = 112.7280 or an additional 12.7280D must be added to the original length to produce sonic conditions for the same M_i and shock location as in part (a).

Problem 16. – Air (γ = 1.4) enters a pipe of diameter 2 cm at a Mach number of M_i = 3.0. A normal shock wave stands in the pipe at a location where the Mach number on the upstream side of the shock is M_1 = 2.0. The Mach number exiting the pipe is M_e = 1.0. For steady, adiabatic, one-dimensional flow in the pipe, i.e., Fanno flow, determine the location of the shock and the total length of the pipe. Assume f = 0.02.

Now from the normal shock relations at M_1 = 2.0 and γ = 1.4, we obtain M_2 = 0.577350. At the various Mach numbers we can determine the corresponding fL_{max}/D ratios, which are needed to locate the shock and determine the pipe length.

Mi = 3.0,	$(fL_{max}/D)_i$ = 0.522159
M_1 = 2.0,	$(fL_{max}/D)_i$ = 0.522159
M_2 = 0.577350,	$(fL_{max}/D)_i$ = 0.522159
M_e = 1.0,	$(fL_{max}/D)_i$ = 0.522159

The shock location is determined from Eq.(9.34)

$$L_s = \left(\frac{D}{f}\right)\left[\left(\frac{fL_{max}}{D}\right)_i - \left(\frac{fL_{max}}{D}\right)_1\right] = \frac{2}{0.02}(0.522159 - 0.304997)$$

$$= 21.7162 \text{ cm}$$

The total length of the pipe can be readily determined from Eq.(9.29)

$$F(M_1) = \left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL_{max}}{D}\right)_1 = \left(\frac{fL}{D}\right) + \left(\frac{fL_{max}}{D}\right)_e - \left(\frac{fL_{max}}{D}\right)_i$$

Hence,

$$L = \frac{D}{f}\left[\left(\frac{fL_{max}}{D}\right)_2 - \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_e + \left(\frac{fL_{max}}{D}\right)_i\right]$$

$$= \frac{2}{0.02}(0.58761 - 0.304997 - 0.0 + 0.522159)$$

$$= 80.5023 \text{ cm}$$

Problem 17. – A rocket nozzle is operating with a stretched out throat, (L = 50 cm and D = 10 cm) as shown in Figure P9.17. If the inlet stagnation conditions are p_{o1} = 1 MPa and T_{o1} = 1500 K, determine the nozzle exit velocity and mass flow for a back pressure of 30 kPa. The diameter of the nozzle at the exit station is the same as at the inlet station: 30 cm. Treat the exhaust gases as perfect, with γ = 1.4 and R = 0.50 kJ/kg · K. Assume isentropic flow in variable-area sections and Fanno flow in constant-area sections with f = 0.22.

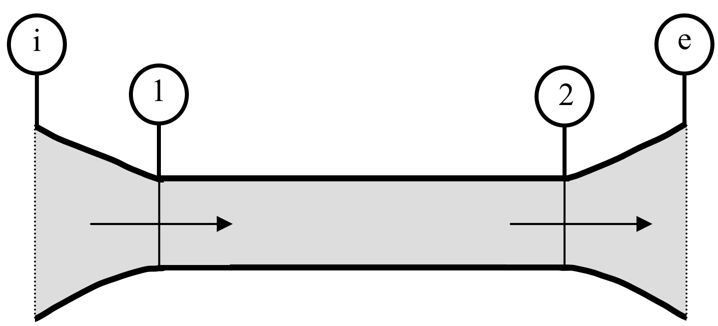


Figure P9.17

Assume the system is choked so that $M_2 = 1$ and $(fL_{max}/D)_2 = 0$. Also, $(p/p_o)_2 = 0.5253$ and $p_2 = p^*$. Now $fL/D = (0.022)(50)/(10) = 0.110 = (fL_{max}/D)_1$. The corresponding Mach number to this value is $M_1 = 0.7637$. At this Mach number from the isentropic relations: $(p/p_o)_1 = 0.67966$ and from the Fanno flow relations: $(p/p^*)_1 = 1.35745$.

Now for an area ratio of 9, from the isentropic relations we find that $M_e = 3.8061$. At this Mach number from the isentropic relations: $(p/p_o)_e = 0.0.008558$. Therefore,

$$\begin{aligned} p_e &= \left(\frac{p_e}{p_{oe}}\right)\left(\frac{p_{o2}}{p_2}\right)\left(\frac{p_2}{p^*}\right)\left(\frac{p^*}{p_1}\right)\left(\frac{p_1}{p_{o1}}\right)p_{o1} \\ &= (0.008558)\left(\frac{1}{0.5283}\right)(1.0)\left(\frac{1}{1.35745}\right)(0.67966)1\,\text{MPa} \\ &= 8.1108\,\text{kPa} \end{aligned}$$

If there is a normal shock at the exit, the pressure ratio across the shock is: $(p_{e2}/p_{e1}) = 16.7337$. Therefore, $p_{e2} = p_b = 135.7230$ kPa. Because this is well above the stated back pressure of 30 kPa, $p_e = 8.1108$ kPa and the flow is further compressed outside the nozzle by oblique shocks.

At $M_e = 3.8061$ from the isentropic relations, $(T/T_o)_e = 0.2566$. Hence,

$$T_e = \frac{T_e}{T_o}T_o = (0.2566)1500 = 384.9000\text{K}$$

$$V_e = M_e a_e = M_e\sqrt{\gamma R T_e} = (3.8061)\sqrt{(1.4)(500(384.9))} = 1975.6189\,\text{m/s}$$

$$\dot{m} = \rho_e A_e V_e = \left(\frac{p_e}{RT_e}\right)A_e V_e = \left[\frac{(8.11076)}{(0.500)384.9}\right]\left(\frac{\pi}{4}0.3^2\right)(1975.6189) = 5.8854\,\text{kg/s}$$

Problem 18. – Air ($\gamma = 1.4$ and $R = 0.287$ kJ/kg · K) flows adiabatically in a tube of circular cross section with an initial Mach number of 0.5, initial $T_1 = 500$ K, and $p_l = 600$ kPa. The tube is to be changed in cross-sectional area so that, taking friction into account, there is no change in the temperature of the stream. Assume the distance between inlet and exit, L, is equal to 100 D_l, with D_l = initial duct diameter; f = 0.02. Find the following:

(a) Mach number M_2
(b) D_2/D_l
(c) Static pressure p_2

(a) Since $T_1 = T_2 = 500$ K and since the flow is adiabatic for which $T_{o1} = T_{o2}$, it therefore follows that $M_1 = M_2 = 0.5$, i.e., the Mach number remains constant as well.

(b) From Eq.(9.42)

$$\frac{dD}{dx} = \frac{\gamma M^2}{4} f$$

Integration yields

$$D_2 - D_1 = f\frac{\gamma M^2}{4} L = 100f\left(\frac{\gamma M^2}{4}\right) D_1$$

Hence,

$$\frac{D_2}{D_1} = 1 + 100f\frac{\gamma M^2}{4} = 1 + (25)(0.02)(1.4)(0.5)^2 = 1.175$$

(c) Since both the static temperature and the Mach number are constant, then so are the speed of sound and the velocity. Accordingly, $p_1A_1 = p_2A_2$. So that

$$p_2 = \frac{A_1}{A_2} p_1 = \left(\frac{D_1}{D_2}\right)^2 p_1 = \left(\frac{1}{1.175}\right)^2 600 = 434.5858\,\text{kPa}$$

Problem 19. – In a rocket nozzle of area ratio 8 to 1, combustion gases (γ = 1.2 and R = 0.50 kJ/kg · K) are expanded from a chamber pressure and temperature of 5 MPa and 2000 K. For a nozzle coefficient C_h equal to 0.96, determine the rocket exhaust velocity in space.

From the isentropic relations with γ = 1.2 and an area ratio, A/A* = 8, the supersonic solution is M = 3.1219. The corresponding temperature ratio T/To = 0.5064. Accordingly, the exit temperature is Te = (0.5064)(2000) = 1012.8 K. The exit velictiy is readily determined as follows:

$$V_e = M_e a_e = M_e\sqrt{\gamma RT_e} = (3.1219)\sqrt{(1.2)(500)(1012.8)} = 2433.6407\,\text{m/s}$$

This is the exit velocity for isentropic flow. The actual exit velocity is related to this speed by

$$C_h = \left(\frac{V_{\text{exit actual}}}{V_{\text{exit isentropic}}}\right)^2$$

Hence,

$$V_{e,actual} = \sqrt{C_h}V_{e,isentropic} = \sqrt{0.96}(2433.6407) = 2384.4712 \text{ m/s}$$

Problem 20. – Air (γ = 1.4) enters a constant-area, insulated duct (Figure 9.9) with a Mach number of 0.50. The duct length is 45 cm; the duct diameter is 3 cm; and the friction coefficient is 0.02. Use Euler's explicit method on a coarse grid containing 11 grid points to determine the Mach number at the duct outlet. Compare the result obtained to the value obtained using Fanno relations.

Exact Solution

From the Fanno relations at M_1 = 0.5, $(fL_{max}/D)_1$ = 1.0691. Also, from the given information fL/D = 0.3. Hence,

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 1.0691 - 0.3 = 0.7691$$

From this value we find that M_2 = 0.542923.

Numerical Solution

Proceeding as in Example problem 9.9, we have

$$\frac{dM}{dx} = F(x, M) = \frac{\gamma M^3[2 + (\gamma - 1)M^2]}{4(1 - M^2)}\frac{f}{D}$$

The same grid in the example is used here in which the duct length is divided into 10 evenly spaced increments, i.e., Δx = 4.5 cm. The computations are straightforward and the results from a spreadsheet program are

pt	x	M_i	$F(x_i,M_i)$	M_{i+1}
1	0.0000	0.5000	0.0008	0.5037
2	4.5000	0.5037	0.0008	0.5075
3	9.0000	0.5075	0.0009	0.5113
4	13.5000	0.5113	0.0009	0.5153
5	18.0000	0.5153	0.0009	0.5195
6	22.5000	0.5195	0.0009	0.5237
7	27.0000	0.5237	0.0010	0.5281
8	31.5000	0.5281	0.0010	0.5326
9	36.0000	0.5326	0.0010	0.5373
10	40.5000	0.5373	0.0011	0.5421
11	45.0000	0.5421		

Hence, the error between the results is (0.5421/0.5429 – 1)100 = -0.1429%.

Problem 21. –If the problem described in the above is solved on finer grids using the first order Euler explicit method, the following results for the exit Mach number are obtained

n	Δx	M_e
11	4.5	0.542147
21	2.25	0.542529
41	1.125	0.542725
81	0.5625	0.542824

Determine the error when compared to the exact value of the exit Mach number is 0.542923. Use Richardson's extrapolation method to obtain improved values. Also, compute the error of these values.

Richardson's extrapolation for Euler's explicit method is given by

$$E_{Euler} = R_2 - \frac{R_2 - R_1}{1-(2)^1} = R_2 + \frac{R_2 - R_1}{1} = 2R_2 - R_1$$

Using this relation, the following table is easily prepared

n	Δx	M_e	% error	M_e (Extr)	% error
11	4.5	0.542147	-0.1429		
21	2.25	0.542529	-0.0726	0.5429115	-0.0022
41	1.125	0.542725	-0.0366	0.5429203	-0.0006
81	0.5625	0.542824	-0.0184	0.5429225	-0.0001

Problem 22. – Heun's predictor-corrector method is 2nd order and the Runge-Kutta method used in this Chapter is 4th order. Obtain an expression for each of these methods that could be used to perform Richardson's extrapolation of results, R2 and R1 that were determined on two grids that differ by a factor of two, i.e., $\Delta x_2 = \Delta x_1/2$.

The extrapolated value, E, in *Richardson's extrapolation method*, is given by

$$E = R_2 - \frac{R_2 - R_1}{1 - \left(\frac{\Delta x_1}{\Delta x_2}\right)^n}$$

where R_1 and R_2 are the values that have been computed using the same method on two grids of known width, say Δx_1 and Δx_2. Also, the accuracy of the method is of order n. For this problem $\Delta x_1 = 2\Delta x_2$. Thus,

Heun's method: n = 2

$$E_{Heun} = R_2 - \frac{R_2 - R_1}{1-(2)^2} = R_2 + \frac{R_2 - R_1}{3} = \frac{4R_2 - R_1}{3}$$

Runge-Kutta: n = 4

$$E_{R-K} = R_2 - \frac{R_2 - R_1}{1-(2)^4} = R_2 + \frac{R_2 - R_1}{15} = \frac{16R_2 - R_1}{15}$$

Problem 23. – An airstream (γ = 1.4, R = 0.287 kJ/kg·K) at Mach 2.0 with a pressure of 100 kPa and a temperature of 270 K, enters a diverging, linear, conical channel with a ratio of exit area to inlet area of 3.0 (see Figure P9.23). The inlet area is 0.008 m^2 and the length is 10.0 cm. The average friction factor is 0.03. Use Heun's predictor-corrector method on a coarse grid of 11 grid points to determine the back pressure, p_b, necessary to produce a normal shock in the channel at 5 cm from the inlet. Assume one-dimensional, steady flow with the air behaving as a perfect gas with constant specific heats. Compare results to the pressure value obtained by assuming isentropic flow except across the normal shock (see Example 4.3). Does friction significantly change the isentropic flow results?

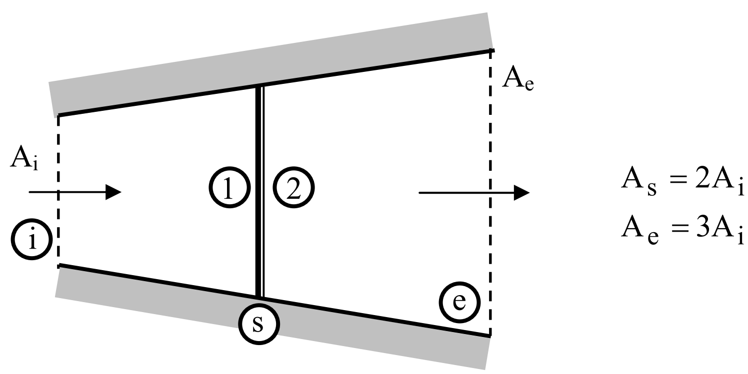


Figure P9.23

Solution Using Fanno Flow, Isentropic Flow and Normal Shock Relations

The inlet diameter is $D_i = \sqrt{4A_i/\pi} = \sqrt{0.032/\pi} = 0.1009\,\text{m}$ and the exit diameter is $D_e = \sqrt{4A_e/\pi} = \sqrt{0.096/\pi} = 0.1749\,\text{m}$. Since the shock is located at L_s = 5cm and since the channel is linear, the diameter at the shock location is

$$D_s = D_i + \frac{D_e - D_i}{L}\frac{L}{2} = \frac{D_e + D_i}{2} = \frac{0.1749 + 0.1009}{2} = 0.1379\,\text{m}$$

Therefore the area at the shock is $A_s = A_1 = \pi D_s^2/4 = 0.0149\ \text{m}^2$. At $M_i = 2.0$, from the isentropic relations with $\gamma = 1.4$,

$$\frac{A_i}{A_1^*} = 1.6875$$

Therefore,

$$\frac{A_1}{A_1^*} = \left(\frac{A_1}{A_i}\right)\left(\frac{A_i}{A_1^*}\right) = \left(\frac{0.0149}{0.008}\right)(1.6875) = 3.1477$$

Using the Newton-Raphson iterative procedure and taking the supersonic root because the flow on the upstream side of the shock must be supersonic, we obtain $M_1 = 2.6882$. Note the Mach number on the downstream side of the shock is found to be 0.4966. With the upstream shock Mach number determined, ratios of properties across the shock can be found from normal shock relations, which are then combined with Eq.(4.21) to give

$$\frac{p_{o2}}{p_{o1}} = 0.4278 = \frac{A_1^*}{A_2^*}$$

or

$$\frac{A_e}{A_2^*} = \left(\frac{A_e}{A_i}\right)\left(\frac{A_i}{A_1^*}\right)\left(\frac{A_1^*}{A_2^*}\right) = (3.0)(1.6875)(0.4278) = 2.1657$$

Again, using the Newton-Raphson procedure, this area ratio produces the following subsonic value at the exit: $M_e = 0.2800$. We can now solve for the exit pressure, p_e:

$$\frac{p_e}{p_i} = \left(\frac{p_e}{p_{o2}}\right)\left(\frac{p_{o2}}{p_{o1}}\right)\left(\frac{p_{o1}}{p_{oi}}\right)\left(\frac{p_{oi}}{p_i}\right) = (0.9470)(0.4278)(1.0)\left(\frac{1}{0.1278}\right) = 3.1700$$

With subsonic flow at the channel exit, the channel back pressure is equal to the exit plane pressure, i.e.,

$p_e = 100(3.1700) = 317.0000\ \text{kPa} = p_b$

Solution Obtained by Solving ODE Using Heun's Method and Normal Shock Relations

The equation which governs the Mach number distribution within the channel when both area variation and frictional effects are considers is

$$\frac{dM}{dx} = F(x,M) = \left\{\frac{M[2+(\gamma-1)M^2]}{2}\right\}\left\{\frac{\left[-\frac{1}{A}\left(\frac{dA}{dx}\right)+\frac{1}{2}\gamma M^2\left(\frac{f}{D}\right)\right]}{(1-M^2)}\right\} \quad \mathbf{(9.52)}$$

Since the nozzle is conical, the local cross sectional area is given by

$A(x), = \pi[D(x)]^2/4 = \pi[D_i + (D_e - D_i)x/L]^2/4,$

Thus the area term in Eq(9.52) can be written as

$$\frac{1}{A(x)}\frac{dA}{dx} = \frac{2}{D(x)}\frac{dD}{dx} = \frac{2\left(\frac{D_e - D_i}{L}\right)}{D(x)}$$

Consequently, Eq.(9.52) becomes

$$\frac{dM}{dx} = F(x,M) = \left\{\frac{M[2+(\gamma-1)M^2]}{2}\right\}\left\{\frac{\left[-\frac{2(D_e - D_i)}{LD}+\frac{1}{2}\gamma M^2\left(\frac{f}{D}\right)\right]}{(1-M^2)}\right\}$$

where $D = D(x) = D_i + (D_e - D_i)x/L$. Heun's method is

Predictor step: $$M_p = M_i + F(x_i, M_i)\Delta x$$

$$\mathbf{(9.49)}$$

Corrector step: $$M_{i+1} = M_i + \left[\frac{F(x_i, M_i) + F(x_{i+1}, M_p)}{2}\right]\Delta x$$

where M_p is the predicted Mach number.

For this problem, we will divide L into 10 pieces of uniform length, i.e., the grid spacing is therefore Δx = 1 cm = 0.01 m.

First, we will assume that f = 0. Inserting M_i and the given information into F(x,M) at x = 0 results in F(0.0,2.0) = 17.5692 and for the grid spacing we can compute the predicted Mach number, i.e., M_p to be 0.2.1757. This is then used to compute

$F(\Delta x, M_p) = 15.4760$. Hence, $M_2 = 2.1602$. The static pressures at each x are computed from

$$p_{i+1} = \frac{D_i^2 M_i}{D_{i+1}^2 M_{i+1}} \left[\frac{2 + (\gamma - 1) M_i^2}{2 + (\gamma - 1) M_{i+1}^2} \right]^{\frac{1}{2}} p_i$$

Across the shock located at x = 0.05 m, the normal shock relations are used, viz.,

$$M_6^2 = \frac{M_5^2 + \dfrac{2}{\gamma - 1}}{\dfrac{2\gamma}{\gamma - 1} M_5^2 - 1}$$

$$\frac{p_6}{p_5} = \frac{2\gamma M_5^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1}$$

The results of the calculations for f = 0 obtained from a spreadsheet program are presented in the following

pt	x	D_i	M_i	$F(x_i,M_i)$	M_p	$F(x_{i+1},M_p)$	M_{i+1}	p_i	p_{i+1}
1	0.0000	0.1009	2.0000	17.5692	2.1757	15.4760	2.1652	100	77.2967
2	0.0100	0.1083	2.1652	15.5185	2.3204	14.0379	2.3130	77.2967	61.3513
3	0.0200	0.1157	2.3130	14.0569	2.4536	12.9317	2.4480	61.3513	49.6999
4	0.0300	0.1231	2.4480	12.9408	2.5774	12.0449	2.5729	49.6999	40.9315
5	0.0400	0.1305	2.5729	12.0494	2.6934	11.3123	2.6897	40.9315	34.1775
6	0.0500	0.1379	2.6897				0.4964	34.1775	282.7671
7	0.0500	0.1379	0.4964	-7.4092	0.4224	-5.1538	0.4336	282.7671	293.2710
8	0.0600	0.1453	0.4336	-5.6371	0.3773	-4.3793	0.3835	293.2710	301.4360
9	0.0700	0.1526	0.3835	-4.4814	0.3387	-3.6142	0.3431	301.4360	307.4827
10	0.0800	0.1600	0.3431	-3.6748	0.3063	-3.0396	0.3095	307.4827	312.0861
11	0.0900	0.1674	0.3095	-3.0789	0.2787	-2.5940	0.2811	312.0861	315.6645
12	0.1000	0.1748	0.2811					315.6645	

The computed exit pressure is 315.6645 kPa, which differs from the value computed from Fanno relations, i.e., 317.0000kPa by –0.4213%. This is very good for the particularly coarse mesh used in the computations.

Repeating the calculations for f = 0.03, we obtain,

pt	x	D_i	M_i	$F(x_i,M_i)$	M_p	$F(x_{i+1},M_p)$	M_{i+1}	p_i	p_{i+1}
1	0.0000	0.1009	2.0000	17.0698	2.1707	14.9773	2.1602	100	77.5617
2	0.0100	0.1083	2.1602	15.0239	2.3105	13.5300	2.3030	77.5617	61.7557
3	0.0200	0.1157	2.3030	13.5524	2.4385	12.4090	2.4328	61.7557	50.1781
4	0.0300	0.1231	2.4328	12.4209	2.5570	11.5053	2.5524	50.1781	41.4465
5	0.0400	0.1305	2.5524	11.5121	2.6676	10.7552	2.6638	41.4465	34.7072
6	0.0500	0.1379	2.6638				0.4985	34.7072	281.5335
7	0.0500	0.1379	0.4985	-7.4504	0.4240	-5.1778	0.4354	281.5335	292.0486
8	0.0600	0.1453	0.4354	-5.6645	0.3787	-4.3986	0.3851	292.0486	300.2143
9	0.0700	0.1526	0.3851	-4.5016	0.3401	-3.6295	0.3444	300.2143	306.2590
10	0.0800	0.1600	0.3444	-3.6906	0.3075	-3.0521	0.3107	306.2590	310.8600
11	0.0900	0.1674	0.3107	-3.0917	0.2798	-2.6045	0.2822	310.8600	314.4358
12	0.1000	0.1748	0.2822					314.4358	

Friction has reduced the exit pressure from 315.6645 kPa to 314.4358 kPa, slightly less than 0.4%. Therefore, an isentropic flow assumption would be acceptable for this problem.

Problem 24. – Helium ($\gamma = 5/3$) flows through a symmetrical, C-D nozzle with a circular cross-section. The shape of the nozzle is given by

$$D = 2\left[1 + \frac{1}{2}\cos\left(\pi\frac{x}{L}\right)\right]D_t$$

The nozzle length is two times the throat diameter, i.e., $L = 2D_t$. Assuming that the nozzle is choked, determine the Mach number distribution for both subsonic and supersonic flow in the diverging portion of the nozzle. Assume that the friction coefficient is 0.4. Use the Method of Beans and the 4th order Runge-Kutta method to solve this problem on a grid in which $\Delta x/L = 0.05$.

Except for the value of γ, this problem is exactly the same as Example 9.10.

First the location of the sonic point must be computed from,

$$\left(\frac{x}{L}\right)_{sp} = -\frac{1}{\pi}\sin^{-1}\left(\frac{\gamma f L}{4\pi D_t}\right)$$

For $\gamma = 1.4$, $f = 0.4$ and $L = 2D_t$, we find that $x_{sp} = 1.0388L$, (the correct angle that appears in the profile equation is186.0907˚). Moreover, $D_{sp}/D_t = 1.0056$ and

$$\left(\frac{dD}{dx}\right)_{sp} = -\frac{\pi}{L}\sin\left(\pi\frac{x_{sp}}{L}\right)D_t = 0.3333\frac{D_t}{L}$$

$$\left(\frac{d^2D}{dx^2}\right)_{sp} = -\frac{\pi^2}{L^2}\cos\left(\pi\frac{x_{sp}}{L}\right)D_t = 9.8139\frac{D_t}{L^2}$$

The slopes that are used to begin the solution at the sonic point must be computed. This is accomplished by solving the following quadratic

$$a\left(\frac{dM}{dx}\right)_{sp}^2 + b\left(\frac{dM}{dx}\right)_{sp} + c = 0$$

where

$$a = \frac{4}{\gamma+1} = 1.6667$$

$$b = \left(\frac{\gamma f}{D}\right)_{sp} = 1.3258/L$$

$$c = -\left(\frac{2}{D}\frac{d^2D}{dx^2}\right)_{sp} + 2\left(\frac{1}{D}\frac{dD}{dx}\right)_{sp}^2 - \left(\frac{\gamma f}{2D^2}\frac{dD}{dx}\right)_{sp} = -19.5176$$

With these coefficients, the two roots for $(dM/dx)_{sp}$ are computed to be

$$\left(\frac{dM}{dx}\right)_{sp} = \frac{-b+\sqrt{b^2-4ac}}{2a} = 3.1922$$

$$\left(\frac{dM}{dx}\right)_{sp} = \frac{-b-\sqrt{b^2-4ac}}{2a} = -4.0761$$

The results of the computations are contained in the following tables

				x_i	x_i	$x_i - \Delta x/2$	$x_i - \Delta x/2$	$x_i - \Delta x$	
				M_i	M_i	$M_i - k_1\Delta x/2$	$M_i - k_2\Delta x/2$	$M_i - k_3\Delta x$	
pt	**x/L**	**D_i/D_t**	**M_i**	**$F(x_i,M_i)$**	**k_1**	**k_2**	**k_3**	**k_4**	**M_{i-1}**
22	1.0338	1.0056	1.0000	3.1922	3.1922	3.0024	3.2560	3.0355	0.8943
21	1.0000	1.0000	0.8943	3.0158	3.0158	2.8504	2.9446	2.7595	0.7496
20	0.9500	1.0123	0.7496	2.7239	2.7239	2.5400	2.5983	2.4077	0.6212
19	0.9000	1.0489	0.6212	2.3930	2.3930	2.2005	2.2439	2.0529	0.5101
18	0.8500	1.1090	0.5101	2.0442	2.0442	1.8566	1.8900	1.7094	0.4163
17	0.8000	1.1910	0.4163	1.7036	1.7036	1.5308	1.5568	1.3946	0.3391

16	0.7500	1.2929	0.3391	1.3905	1.3905	1.2387	1.2587	1.1192	0.2765
15	0.7000	1.4122	0.2765	1.1162	1.1162	0.9881	1.0033	0.8874	0.2266
14	0.6500	1.5460	0.2266	0.8853	0.8853	0.7804	0.7917	0.6979	0.1872
13	0.6000	1.6910	0.1872	0.6964	0.6964	0.6125	0.6207	0.5463	0.1563
12	0.5500	1.8436	0.1563	0.5452	0.5452	0.4791	0.4850	0.4265	0.1322
11	0.5000	2.0000	0.1322	0.4258	0.4258	0.3741	0.3783	0.3326	0.1133
10	0.4500	2.1564	0.1133	0.3322	0.3322	0.2918	0.2947	0.2591	0.0986
9	0.4000	2.3090	0.0986	0.2588	0.2588	0.2272	0.2292	0.2012	0.0872
8	0.3500	2.4540	0.0872	0.2010	0.2010	0.1761	0.1775	0.1553	0.0783
7	0.3000	2.5878	0.0783	0.1552	0.1552	0.1353	0.1362	0.1183	0.0715
6	0.2500	2.7071	0.0715	0.1182	0.1182	0.1020	0.1026	0.0879	0.0664
5	0.2000	2.8090	0.0664	0.0878	0.0878	0.0743	0.0747	0.0622	0.0626
4	0.1500	2.8910	0.0626	0.0622	0.0622	0.0505	0.0508	0.0398	0.0601
3	0.1000	2.9511	0.0601	0.0398	0.0398	0.0293	0.0295	0.0194	0.0586
2	0.0500	2.9877	0.0586	0.0194	0.0194	0.0096	0.0097	0.0000	0.0581
1	0.0000	3.0000	0.0581						

				x_i	x_i	$x_i + \Delta x/2$	$x_i + \Delta x/2$	$x_i + \Delta x$	
Subsonic decelerating flow				M_i	M_i	$M_i + k_1\Delta x/2$	$M_i + k_2\Delta x/2$	$M_i + k_3\Delta x$	
pt	**x/L**	**D_i/D_t**	**M_i**	**$F(x_i,M_i)$**	**k_1**	**k_2**	**k_3**	**k_4**	**M_{i+1}**
23	1.0338	1.0056	1.0000	-4.0761	-4.0761	-3.5747	-4.0292	-3.5432	0.9385
24	1.0500	1.0123	0.9385	-3.9813	-3.9813	-3.3956	-3.7328	-3.3190	0.7589
25	1.1000	1.0489	0.7589	-3.1672	-3.1672	-2.8402	-2.9343	-2.6295	0.6143
26	1.1500	1.1090	0.6143	-2.6052	-2.6052	-2.3246	-2.3823	-2.1245	0.4965
27	1.2000	1.1910	0.4965	-2.1125	-2.1125	-1.8753	-1.9145	-1.6986	0.4015
28	1.2500	1.2929	0.4015	-1.6916	-1.6916	-1.4946	-1.5223	-1.3448	0.3260
29	1.3000	1.4122	0.3260	-1.3403	-1.3403	-1.1799	-1.1998	-1.0566	0.2663
30	1.3500	1.5460	0.2663	-1.0536	-1.0536	-0.9254	-0.9396	-0.8259	0.2196
31	1.4000	1.6910	0.2196	-0.8239	-0.8239	-0.7228	-0.7330	-0.6438	0.1831
32	1.4500	1.8436	0.1831	-0.6425	-0.6425	-0.5636	-0.5707	-0.5013	0.1546
33	1.5000	2.0000	0.1546	-0.5004	-0.5004	-0.4392	-0.4441	-0.3902	0.1325
34	1.5500	2.1564	0.1325	-0.3897	-0.3897	-0.3421	-0.3455	-0.3036	0.1153
35	1.6000	2.3090	0.1153	-0.3032	-0.3032	-0.2661	-0.2685	-0.2356	0.1019
36	1.6500	2.4540	0.1019	-0.2353	-0.2353	-0.2061	-0.2077	-0.1817	0.0915
37	1.7000	2.5878	0.0915	-0.1815	-0.1815	-0.1582	-0.1593	-0.1383	0.0835
38	1.7500	2.7071	0.0835	-0.1382	-0.1382	-0.1193	-0.1200	-0.1027	0.0775
39	1.8000	2.8090	0.0775	-0.1027	-0.1027	-0.0868	-0.0873	-0.0726	0.0732
40	1.8500	2.8910	0.0732	-0.0726	-0.0726	-0.0590	-0.0593	-0.0464	0.0702
41	1.9000	2.9511	0.0702	-0.0464	-0.0464	-0.0342	-0.0344	-0.0226	0.0685
42	1.9500	2.9877	0.0685	-0.0226	-0.0226	-0.0112	-0.0112	0.0001	0.0679
43	2.0000	3.0000	0.0679						

				x	x_i	$x_i + \Delta x/2$	$x_i + \Delta x/2$	$x_i + \Delta x$	
Supersonic accelerating flow				M	M_i	$M_i + k_1\Delta x/2$	$M_i + k_2\Delta x/2$	$M_i + k_3\Delta x$	
pt	**x/L**	**D_i/D_t**	**M_i**	**$F(x_i,M_i)$**	**k_1**	**k_2**	**k_3**	**k_4**	**M_{i+1}**
23	1.0338	1.0056	1.0000	3.1922	3.1922	3.3531	3.1553	3.2836	1.0525
24	1.0500	1.0123	1.0525	3.2282	3.2282	3.3841	3.2681	3.3517	1.2182
25	1.1000	1.0489	1.2182	3.4150	3.4150	3.4503	3.4388	3.4513	1.3902
26	1.1500	1.1090	1.3902	3.4561	3.4561	3.4439	3.4463	3.4104	1.5623
27	1.2000	1.1910	1.5623	3.4113	3.4113	3.3534	3.3617	3.2830	1.7300
28	1.2500	1.2929	1.7300	3.2829	3.2829	3.1857	3.1965	3.0825	1.8894
29	1.3000	1.4122	1.8894	3.0820	3.0820	2.9533	2.9651	2.8234	2.0373
30	1.3500	1.5460	2.0373	2.8227	2.8227	2.6699	2.6819	2.5196	2.1710
31	1.4000	1.6910	2.1710	2.5189	2.5189	2.3486	2.3604	2.1836	2.2887
32	1.4500	1.8436	2.2887	2.1830	2.1830	2.0008	2.0123	1.8260	2.3890
33	1.5000	2.0000	2.3890	1.8254	1.8254	1.6358	1.6469	1.4551	2.4710
34	1.5500	2.1564	2.4710	1.4545	1.4545	1.2610	1.2717	1.0774	2.5343
35	1.6000	2.3090	2.5343	1.0768	1.0768	0.8820	0.8922	0.6977	2.5787
36	1.6500	2.4540	2.5787	0.6972	0.6972	0.5030	0.5128	0.3197	2.6041
37	1.7000	2.5878	2.6041	0.3192	0.3192	0.1271	0.1363	-0.0543	2.6107
38	1.7500	2.7071	2.6107	-0.0547	-0.0547	-0.2439	-0.2351	-0.4225	2.5987
39	1.8000	2.8090	2.5987	-0.4229	-0.4229	-0.6087	-0.6005	-0.7845	2.5685
40	1.8500	2.8910	2.5685	-0.7848	-0.7848	-0.9673	-0.9595	-1.1404	2.5203
41	1.9000	2.9511	2.5203	-1.1407	-1.1407	-1.3204	-1.3131	-1.4917	2.4545
42	1.9500	2.9877	2.4545	-1.4920	-1.4920	-1.6702	-1.6635	-1.8419	2.3712
43	2.0000	3.0000	2.3712						

Chapter Ten

FLOW WITH HEAT ADDITION OR HEAT LOSS

Problem 1. – Draw the T-s diagram for the flow of a gas with γ = 1.4 in a constant diameter pipe with heat addition or loss. The reference Mach number, M_1, for the flow is 3.0.

This is a companion to Example 10.2. In that problem the reference state is the same as given here, however, γ = 1.3. To draw the Rayleigh line for the given reference state we have

$$\frac{T}{T_1} = \frac{M^2}{M_1^2}\frac{\left(1+\gamma M_1^2\right)^2}{\left(1+\gamma M^2\right)^2}$$

and

$$\frac{s-s_1}{c_p} = \ln\left[\left(\frac{M}{M_1}\right)^2\left(\frac{1+\gamma M_1^2}{1+\gamma M^2}\right)^{\frac{(\gamma+1)}{\gamma}}\right]$$

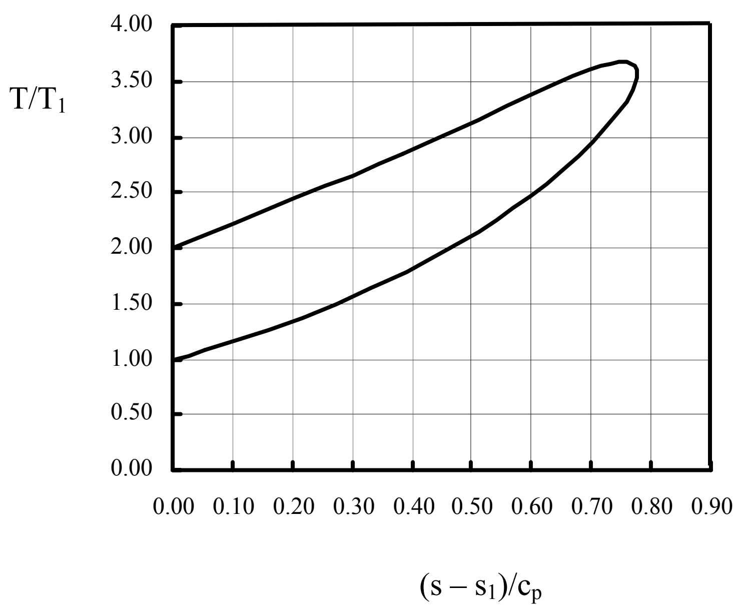
It should be noted that the entropy change is zero for M = M_1 and therefore at T = T_1. The second value is determined by setting the argument of the natural logarithm to 1 and solving the nonlinear equation using the Newton-Raphson method. The function that is solved and its derivative are

$$F(M) = cM^b - \gamma M^2 - 1 = 0$$

$$\frac{dF}{dM} = bcM^{b-1} - 2\gamma M$$

where $c = \left(1+\gamma M_1^2\right)/M_1^b$ and $b = 2\gamma/(\gamma+1)$. For M_1 = 3.0 and γ = 1.4 the solution procedure yields M = 0.37307. The calculations to draw the Rayleigh line were performed within a spreadsheet program and the results are contained in the following table and figure

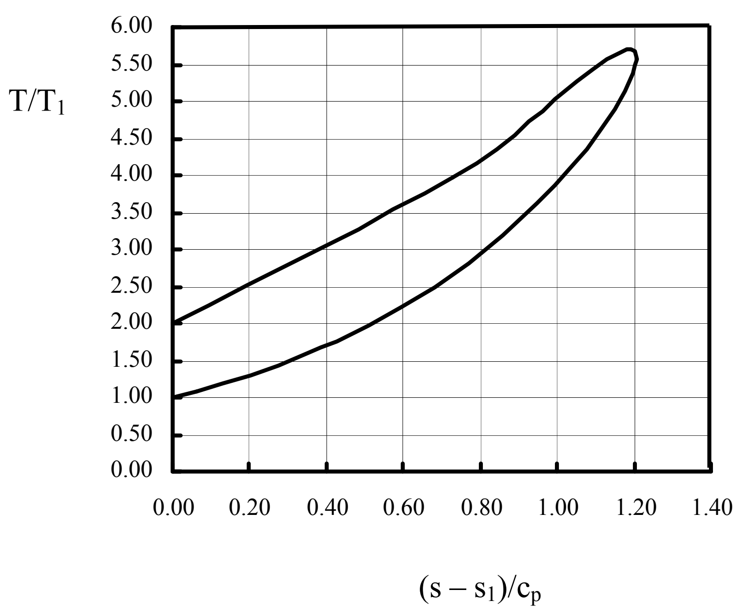
M	$\Delta s/c_p$	T/T_1
0.3731	0.0000	2.0035
0.50	0.3861	2.8429
0.64	0.6026	3.3878
0.77	0.7167	3.6356
0.90	0.7667	3.6562
1.03	0.7757	3.5302
1.16	0.7581	3.3231
1.29	0.7236	3.0798
1.42	0.6782	2.8281
1.56	0.6259	2.5838
1.69	0.5696	2.3550
1.82	0.5111	2.1453
1.95	0.4516	1.9554
2.08	0.3919	1.7847
2.21	0.3327	1.6318
2.34	0.2742	1.4952
2.47	0.2168	1.3732
2.61	0.1606	1.2641
2.74	0.1057	1.1664
2.87	0.0522	1.0788
3.00	0.0000	1.0000



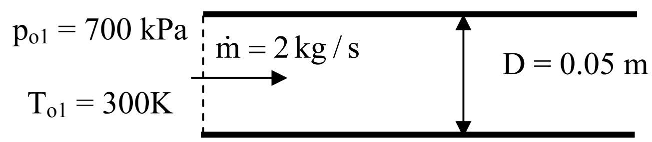
Problem 2. – Draw the T-s diagram for the flow of a gas with $\gamma = 1.3$ in a constant diameter pipe with heat addition or loss. The reference Mach number, M_1, for the flow is 4.0.

This is a companion to Example 10.2 and problem 1. In this problem the reference state differs from these previous problems, however, $\gamma = 1.3$. For $M_1 = 4.0$ and $\gamma = 1.4$ the solution procedure yields M = 0.2864. The calculations to draw the Rayleigh line were performed within a spreadsheet program and the results are contained in the following table and figure

M	$\Delta s/c_p$	T/T_1
0.2864	0.0000	1.9894
0.47	0.7287	3.9795
0.66	1.0527	5.2640
0.84	1.1809	5.7033
1.03	1.2057	5.5685
1.21	1.1742	5.1463
1.40	1.1123	4.6233
1.59	1.0341	4.0973
1.77	0.9481	3.6116
1.96	0.8588	3.1812
2.14	0.7691	2.8073
2.33	0.6806	2.4855
2.51	0.5941	2.2093
2.70	0.5102	1.9724
2.89	0.4290	1.7685
3.07	0.3507	1.5926
3.26	0.2752	1.4401
3.44	0.2026	1.3075
3.63	0.1325	1.1915
3.81	0.0651	1.0897
4.00	0.0000	1.0000



Problem 3. – Air (γ = 1.4 and R = 0.287 kJ/kg · K) flows in a constant-area duct of 5-cm diameter at a rate of 2 kg/s. If the inlet stagnation pressure and temperature are, respectively, 700 kPa and 300 K, plot T versus s for Rayleigh line flow. For the same inlet conditions and mass flow rate, plot a T-s diagram for Fanno flow. From the points of intersection of Rayleigh and Fanno lines, show the states on either side of a normal shock. Assume the air to behave as a perfect gas with constant specific heats.



The flow rate may be written as

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1} = \left[\frac{\left(\frac{p_1}{p_{o1}}\right) p_{o1}}{R\left(\frac{T_1}{T_{o1}}\right) T_{o1}}\right]\left(\frac{\pi}{4}D^2\right) M_1 \sqrt{\gamma R\left(\frac{T_1}{T_{o1}}\right) T_{o1}}$$

So

$$\frac{\dot{m}}{1000(p_{o1})\left(\left(\frac{\pi}{4}D^2\right)\right)}\sqrt{\frac{RT_{o1}}{\gamma}} = c = \frac{M_1\sqrt{\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]}}{\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}}}$$

This is a nonlinear algebraic equation that can be solved by the Newton-Raphson method. The following is a table that presents the iterations to determine the two Mach numbers by this approach. The function that is to be solved to determine the Mach numbers and its derivative are

$$f(M) = \frac{\gamma-1}{2}M^4 + M^2 - c^2\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{2\gamma}{\gamma-1}} = 0$$

$$\frac{df}{dM}(M) = 2(\gamma-1)M^3 + 2M - 2\gamma c^2 M\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{\gamma-1}}$$

where

$$c = \frac{\dot{m}}{1000(p_{o1})\left(\frac{\pi}{4}D^2\right)}\sqrt{\frac{RT_{o1}}{\gamma}} = \frac{2}{(250\pi)(700)(0.05)^2}\sqrt{\frac{(287)(300)}{1.4}} = 0.3609$$

The calculations performed on a spreadsheet are as follows:

n	M_{old}	f(M)	df/dm	M_{new}		n	M_{old}	f(M)	df/dm	M_{new}
1	0.2000	-0.0974	0.3299	0.4951		1	3.0000	-150.5055	-499.5166	2.6987
2	0.4951	0.0751	0.8468	0.4064		2	2.6987	-52.4180	-195.1466	2.4301
3	0.4064	0.0071	0.6864	0.3960		3	2.4301	-17.6948	-79.0430	2.2062
4	0.3960	0.0001	0.6679	0.3959		4	2.2062	-5.5758	-34.5185	2.0447
5	0.3959	1.85E-08	0.6677	0.3959		5	2.0447	-1.4866	-17.6427	1.9604
6	0.3959	8.33E-16	0.6677	0.3959		6	1.9604	-0.2525	-11.9319	1.9393
7	0.3959	-2.22E-16	0.6677	0.3959		7	1.9393	-0.0127	-10.7471	1.9381
8	0.3959	0.00E+00	0.6677	0.3959		8	1.9381	0.0000	-10.6836	1.9381
9	0.3959	0.00E+00	0.6677	0.3959		9	1.9381	-3.31E-10	-10.6834	1.9381
10	0.3959	0.00E+00	0.6677	0.3959		10	1.9381	0.00E+00	-10.6834	1.9381
				Answer						Answer

Thus, two possible Mach numbers are obtained-one subsonic and the other supersonic. Since we seek to show that the intersection of the Rayleigh and Fanno lines correspond to the states on either side of a normal shock, only the supersonic result needs to be considered, i.e., the reference Mach number, $M_1 = 1.9381$.

Rayleigh line: Following the procedure of Example 10.2 we may write

$$\frac{T}{T_1} = \frac{M^2}{M_1^2}\frac{\left(1+\gamma M_1^2\right)^2}{\left(1+\gamma M^2\right)^2}$$

and

$$\frac{s-s_1}{c_p} = \ln\left\{\left[\frac{M^2}{M_1^2}\frac{\left(1+\gamma M_1^2\right)^2}{\left(1+\gamma M^2\right)^2}\right]\left[\frac{\left(1+\gamma M^2\right)}{\left(1+\gamma M_1^2\right)}\right]^{\frac{(\gamma-1)}{\gamma}}\right\} = \ln\left[\left(\frac{M}{M_1}\right)^2\left(\frac{1+\gamma M_1^2}{1+\gamma M^2}\right)^{\frac{(\gamma+1)}{\gamma}}\right]$$

Incorporating these into a spreadsheet program results in the following table of data

M	$\Delta s/c_p$	T/T_1
0.5393	0.0000	1.5317
0.60	0.0992	1.6597
0.70	0.2117	1.7976
0.80	0.2775	1.8566
0.90	0.3104	1.8549
1.00	0.3197	1.8105

1.10	0.3123	1.7386
1.20	0.2927	1.6509
1.30	0.2646	1.5555
1.40	0.2304	1.4582
1.50	0.1919	1.3624
1.60	0.1504	1.2705
1.70	0.1071	1.1836
1.80	0.0625	1.1025
1.90	0.0173	1.0272
1.9381	0.0000	1.0000

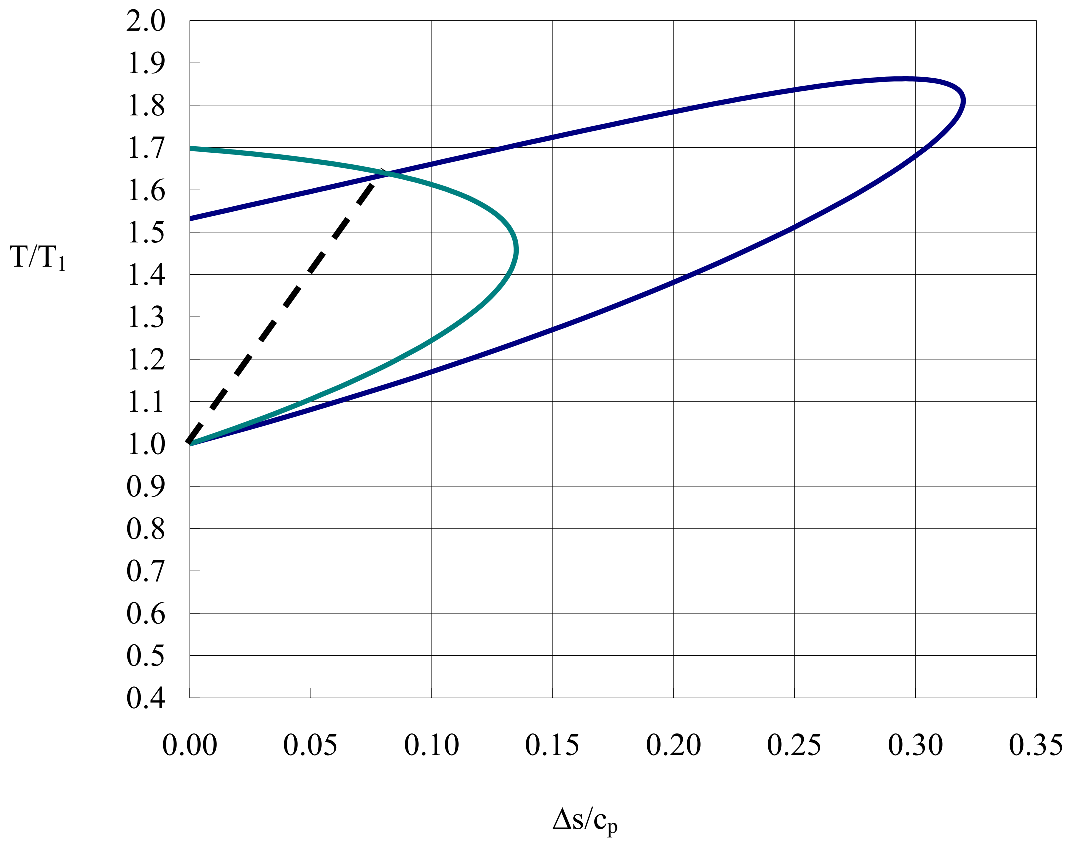
Fanno line: Following the procedure of Example 9.1 we may write

$$\frac{T_o}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 \qquad \frac{T}{T_1} = \frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M^2} \qquad \frac{s - s_1}{c_p} = \frac{1}{\gamma} \ln\left(\frac{T}{T_1}\right) + \frac{\gamma - 1}{2\gamma} \ln\left(\frac{\frac{T_o}{T_1} - \frac{T}{T_1}}{\frac{T_o}{T_1} - 1}\right)$$

Incorporating these into a spreadsheet program results in the following table of data

M	**$\Delta s/c_p$**	**T/T_1**
0.3959	0.0000	1.6980
0.40	0.0024	1.6969
0.50	0.0514	1.6679
0.60	0.0857	1.6336
0.70	0.1092	1.5949
0.80	0.1242	1.5525
0.90	0.1324	1.5071
1.00	0.1349	1.4594
1.10	0.1327	1.4100
1.20	0.1264	1.3597
1.30	0.1166	1.3089
1.40	0.1039	1.2581
1.50	0.0886	1.2078
1.60	0.0711	1.1582
1.70	0.0518	1.1098
1.80	0.0310	1.0626
1.90	0.0088	1.0170
1.9381	0.0000	1.0000

Plotting the data in a single figure results in



To obtain the intersection point of the two lines we must obtain the value of T/T_1 from an equation that includes Eq.(10.17) and Eq.(9.7), i.e.,

$$\ln\frac{T}{T_1}-\frac{\gamma-1}{\gamma}\ln\left[\frac{\left(1+\gamma M_1^2\right)+\sqrt{\left(1+\gamma M_1^2\right)^2-4\gamma M_1^2\left(T/T_1\right)}}{2}\right]-\frac{1}{\gamma}\ln\left(\frac{T}{T_1}\right)$$

$$+\frac{\gamma-1}{2\gamma}\ln\left(\frac{1+\frac{\gamma-1}{2}M_1^2-\frac{T}{T_1}}{\frac{\gamma-1}{2}M_1^2}\right)=0$$

For the given value of $M_1 = 1.9381$ we must iteratively solve to this nonlinear equation to obtain $T/T_1 = 1.6378$. Using the normal shock relations at M1 = 1.9381, we find that $T_2/T_1 = 1.6378$.

Problem 4. – Air ($\gamma = 1.4$, $R = 0.287$ kJ/kg · K and $c_p = 1.004$ kJ/ kg · K) flows in a constant-area duct of diameter 1.5 cm with a velocity of 100 m/s, static temperature of 320 K, and static pressure of 200 kPa. Determine the rate of heat input to the flow necessary to choke the duct. Assume Rayleigh line flow; express your answer in kilowatts. Assume the air to behave as a perfect gas with constant specific heats.

The Mach number at the initial station is

$$M_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt{\gamma RT_1}} = \frac{100}{\sqrt{(1.4)(287)(320)}} = 0.2789$$

At this Mach number from the isentropic relation $(T/T_o)_1 = 0.9847$. Thus, $T_{o1} = T_1/0.9847 = 320/0.9847 = 324.9721K$. Now using the initial Mach number in the Rayleigh relation we find that $T_{o1}/T_o^* = 0.3084$. Hence, $T_o^* = 324.9721/0.3084 = 1053.7356K$.

The flow rate is given by

$$\dot{m} = \rho_1 AV_1 = \left(\frac{p_1}{RT_1}\right) AV_1 = \left[\frac{(200)}{(0.287)(320)}\right]\left(\frac{\pi}{4}0.015^2\right)100 = 0.03848 kg/s$$

The heat transfer rate for choked flow is

$$\dot{q} = \dot{m}c_p\left(T_o^* - T_{o1}\right) = (0.03848)(1.004)(1053.7356 - 324.9721) = 28.1573 kJ/s$$

$$= 28.1573 kW$$

Problem 5. – Air ($\gamma = 1.4$, $R = 0.287$ kJ/kg · K and $c_p = 1.004$ kJ/ kg · K) flows in a constant-area duct of 10 cm diameter at a rate of 0.5 kg/s. The inlet stagnation pressure is 100 kPa; inlet stagnation temperature is 35°C. Find the following:

(a) Two possible values of inlet Mach number.

(b) For each inlet Mach number of part (a), determine the heat addition rate in kilowatts necessary to choke the duct.

$$\dot{m} = \rho_1 AV_1 = \left(\frac{p_1}{RT_1}\right) AM_1\sqrt{\gamma RT_1} = \left[\frac{\left(\frac{p_1}{p_{o1}}\right)p_{o1}}{R\left(\frac{T_1}{T_{o1}}\right)T_{o1}}\right]\left(\frac{\pi}{4}D^2\right)M_1\sqrt{\gamma R\left(\frac{T_1}{T_{o1}}\right)T_{o1}}$$

$$\dot{m} = 0.5\text{kg/s} = \left[\frac{\left(\frac{p_1}{p_{o1}}\right)100}{0.287\left(\frac{T_1}{T_{o1}}\right)308}\right]\left(\frac{\pi}{4}0.1^2\right)M_1\sqrt{(1.4)(287)\left(\frac{T_1}{T_{o1}}\right)(308)}$$

$$= 3.1256\frac{\left(\frac{p_1}{p_{o1}}\right)M_1}{\sqrt{\frac{T_1}{T_{o1}}}} = 3.1256\frac{M_1\sqrt{\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]}}{\left[1+\left(\frac{\gamma-1}{2}\right)M_1^2\right]^{\frac{\gamma}{\gamma-1}}}$$

This is a nonlinear algebraic equation that can be solved by the Newton-Raphson method. The following is a table that presents the iterations to determine the two Mach numbers by this approach. The function that is to be solved to determine the Mach numbers and its derivative are

$$f(M) = \frac{\gamma-1}{2}M^4 + M^2 - c^2\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{2\gamma}{\gamma-1}} = 0$$

$$\frac{df}{dM}(M) = 2(\gamma-1)M^3 + 2M - 2\gamma c^2 M\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{\gamma-1}}$$

where c = 0.5/(3.1256) = 0.1600.

Iteration	M_{old}	f(M)	df/dm	M_{new}		Iteration	M_{old}	f(M)	df/dm	M_{new}
1	0.4000	0.1332	0.8166	0.2369		1	3.0000	-9.3419	-76.0257	2.8771
2	0.2369	0.0291	0.4662	0.1745		2	2.8771	-1.8595	-47.5194	2.8380
3	0.1745	0.0039	0.3404	0.1630		3	2.8380	-0.1408	-40.4679	2.8345
4	0.1630	0.0001	0.3173	0.1626		4	2.8345	-0.0010	-39.8816	2.8345
5	0.1626	1.75E-07	0.3165	0.1625		5	2.8345	-5.5E-08	-39.8773	2.8345
6	0.1625	3.04E-13	0.3165	0.1625		6	2.8345	0.00E+00	-39.8773	2.8345
7	0.1625	0.00E+00	0.3165	0.1625		7	2.8345	0.00E+00	-39.8773	2.8345

(a) So M_1 = 0.1625 and 2.8345 are the two possible initial Mach numbers for the given conditions.
(b) At M_1 = 0.1625, from the Rayleigh relation we find that T_{o1}/T_o^* = 0.1185. Hence, T_o^* = 308/0.1185 = 2599.1561K.

The heat transfer rate for choked flow is

$$\dot{q} = \dot{m}c_p\left(T_o^* - T_{o1}\right) = (0.5)(1.004)(2599.1561 - 308) = 1150.1604\text{kW}$$

At $M_1 = 2.8345$, from the Rayleigh relation we find that $T_{o1}/T_o^* = 0.6702$. Hence, $T_o^* = 308/0.6702 = 459.5643$K.

The heat transfer rate for choked flow is

$$\dot{q} = \dot{m}c_p\left(T_o^* - T_{o1}\right) = (0.5)(1.004)(459.5643 - 308) = 76.0853\text{kW}$$

Problem 6. – A. supersonic flow at $p_o = 1.0$ MPa and $T_o = 1000$ K enters a 5 cm diameter duct at Mach 1.8. Heat is added to the flow via a chemical reaction taking place inside the duct. Determine the heat transfer rate in kilowatts necessary to choke the duct. Assume the air ($\gamma = 1.4$, $R = 0.287$ kJ/kg · K and $c_p = 1.004$ kJ/ kg · K) to behave as a perfect gas with constant specific heats; neglect changes in the composition of the gas stream due to the chemical reaction.

At $M_1 = 1.8$ from the Rayleigh relation we find that $T_{o1}/T_o^* = 0.8363$. Hence, $T_o^* = 1000/0.8363 = 1195.7432$K. Also from the isentropic relations at this Mach number $(T/T_o)_1 = 0.6068$ and $(p/p_o)_1 = 0.1740$. Thus, $T_1 = (T_{o1})(0.6068) = 606.8$ K and $p_1 = (p_{o1})(0.1740) = 174.0$ kPa.

The flow rate is given by

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1} = \left[\frac{(174)}{(0.287)(606.8)}\right]\left(\frac{\pi}{4}0.05^2\right)1.8\sqrt{(1.4)(287)(606.8)}$$
$$= 1.7436\text{kg/s}$$

The heat transfer rate for choked flow is

$$\dot{q} = \dot{m}c_p\left(T_o^* - T_{o1}\right) = (1.7436)(1.004)(1195.7432 - 1000) = 342.6630\text{kW}$$

Problem 7. – Heat is added to airflow ($\gamma = 1.4$ and $R = 0.287$ kJ/kg · K) in a constant-area duct at the rate of 30 kJ/m. If flow enters at Mach 0.20, $T_1 = 300$ K, and $p_1 = 100$ kPa, determine $M(x)$, $p(x)$, $T(x)$, and $p_o(x)$.

From isentropic relations at $M_1 = 0.2$, $T_1/T_{o1} = 0.9921$. Accordingly,

$$T_{o1} = \frac{T_{o1}}{T_1} T_1 = \left(\frac{1}{0.9921}\right) 300 = 302.3889\,\text{K}$$

From the Rayleigh relations at $M_1 = 0.2$, $T_{o1}/T^*_o = 0.1736$. Hence,

$$T_o^* = \frac{T_o^*}{T_{o1}} T_{o1} = \left(\frac{1}{0.1736}\right) 302.3889 = 1741.8714\,\text{K}$$

The maximum length that the pipe may have without affecting the mass flow rate is obtained when $T_{oe} = T^*_o$. Therefore from Eq.(10.10),

$$q_{max} = q' L_{max} = c_p \left(T_o^* - T_{o1}\right)$$

Solving for the maximum length

$$L_{max} = \frac{c_p\left(T_o^* - T_{o1}\right)}{q'} = \frac{\gamma R}{\gamma - 1} \frac{T_o^* - T_{o1}}{q'} = \frac{(1.4)(287)}{0.4}\left(\frac{1741.8714 - 302.3889}{30{,}000}\right)$$

$$= 48.1987\,\text{m}$$

The distribution of $T_o(x)$ can also be determined from Eq.(10.10)

$$T_o(x) = T_{o1} + \frac{x}{L_{max}}\left(T_o^* - T_{o1}\right) = 302.3889 + 29.8656x$$

Hence,

$$\frac{T_o(x)}{T_o^*} = \frac{302.3889 + 29.8656x}{1741.8714} = 0.1736 + 0.017146x$$

Now for a given x, Eq.(10.14) may be used to determine M(x), i.e., letting $t = T_o/T_o^*$, $b = 1 - \gamma(t - 1)$ and $a = 1 + \gamma^{2(}t - 1)$ we have

$$M^2 = \frac{b - \sqrt{b^2 - at}}{a}$$

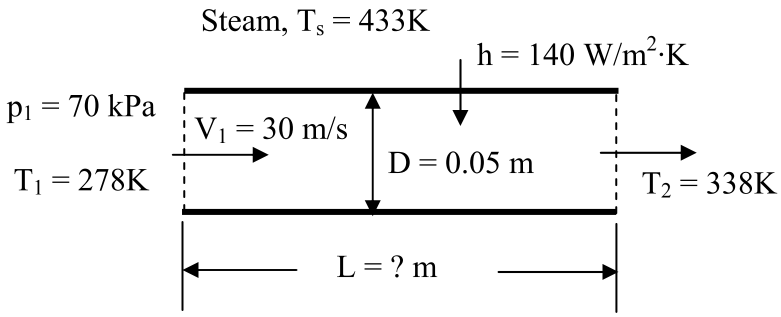
Note the − sign in front of the radical is used to obtain a subsonic Mach number. With this Mach number it is an easy matter to obtain the static pressure and temperature distributions from Eqs.(10.8) and (10.9), respectively. The stagnation pressure distribution can be obtained from

$$p_o(x) = \frac{p_o(x)}{p(x)} p(x) = p(x)\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

The following table contains the results obtained from a simple spreadsheet program

x	T_o/T_o^*	M	p(x)	T(x)	$p_o(x)$
0.00	0.1736	0.2000	100.0000	300.0000	102.8281
2.50	0.2165	0.2264	98.5268	373.3398	102.1090
5.00	0.2593	0.2514	97.0143	446.2152	101.3754
7.50	0.3022	0.2755	95.4574	518.6717	100.6256
10.00	0.3451	0.2990	93.8519	590.6699	99.8584
12.50	0.3879	0.3223	92.1929	662.1641	99.0723
15.00	0.4308	0.3456	90.4747	733.1003	98.2658
17.50	0.4737	0.3690	88.6906	803.4142	97.4371
20.00	0.5165	0.3929	86.8322	873.0279	96.5839
22.50	0.5594	0.4175	84.8894	941.8461	95.7037
25.00	0.6023	0.4429	82.8495	1,009.7494	94.7936
27.50	0.6451	0.4695	80.6965	1,076.5853	93.8497
30.00	0.6880	0.4977	78.4090	1,142.1534	92.8674
32.50	0.7308	0.5280	75.9582	1,206.1816	91.8409
35.00	0.7737	0.5610	73.3031	1,268.2839	90.7624
37.50	0.8166	0.5978	70.3821	1,327.8795	89.6212
40.00	0.8594	0.6403	67.0947	1,384.0193	88.4021
42.50	0.9023	0.6915	63.2538	1,434.9416	87.0806
45.00	0.9452	0.7595	58.4241	1,476.5406	85.6122
47.50	0.9880	0.8788	50.7381	1,491.2081	83.8824
48.1987	1.0000	1.0000	44.0000	1,452.0000	83.2889

Problem 8. – An airstream (γ = 1.4 and R = 0.287 kJ/kg · K) passing through a 5-cm-diameter, thin-walled tube is to be heated by high-pressure steam condensing on the outer surface of the tube at 160°C. The overall heat transfer coefficient between steam and air can be assumed to be 140 W/m^2·K, with the air entering at 30 m/s, 70 kPa, and 5°C. The air is to be heated to 65°C. Determine the tube length required. Assuming Rayleigh line flow, calculate the static pressure change due to heat addition. Also, for the same inlet conditions, calculate the pressure drop due to friction, assuming Fanno flow in the duct with f = 0.018. To obtain an approximation to the overall pressure drop in this heat exchanger, add the two results.



Because the wall of the pipe is thin, assume the wall has a single temperature, i.e., is radially lumped. Also, since the steam is condensing on the outside of the pipe, the pipe temperature may be assumed to be equal to the T_s. The air is treated as a perfect gas, so

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)287}{1.4 - 1} = 1004.5\,\mathrm{J/kg\cdot K}$$

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)(287)(278)} = 334.2161\,\mathrm{m/s}$$

$$M_1 = \frac{V_1}{a_1} = \frac{30}{334.2161} = 0.08976$$

At this Mach number we find from the Rayleigh relations for static pressure and temperature and an isentropic flow temperature relation that

$$\left(\frac{p}{p^*}\right)_1 = 2.3732$$

$$\left(\frac{T}{T^*}\right)_1 = 0.04538$$

$$\left(\frac{T}{T_o}\right)_1 = 0.9984$$

Now

$$\frac{T_2}{T^*} = \frac{T_2}{T_1}\frac{T_1}{T^*} = \frac{338}{278}0.04538 = 0.05517$$

From this temperature ratio, we may use the Rayleigh relations to find that $M_2 = 0.09922$. At this Mach number, we find that $(T/T_o)_2 = 0.9980$ and that $(p/p^*)_2 = 2.3674$.

The mass flow rate in the pipe may be computed as follows

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)\left(\frac{\pi}{4}D^2\right)V_1 = \frac{70}{(0.287)(278)}\frac{\pi}{4}(0.05)^2(30) = 0.05168\,\mathrm{kg/s}$$

From an energy balance on a differential control volume

$$\dot{m}c_p dT_o = \delta q = \bar{h}dA(T_w - T_o) = \bar{h}\pi D(T_s - T_o)dx$$

Rearranging

$$-\frac{dT_o}{T_o - T_s} = \frac{\bar{h}\pi D}{\dot{m}c_p}dx$$

Integrating along the length of the pipe gives

$$\ln\frac{T_{o1} - T_s}{T_{o2} - T_s} = \frac{\pi D\bar{h}}{\dot{m}c_p}L$$

To1 and To2 are computed from the given static temperatures and the values of the static to total temperature ratios determined above

$$T_{o1} = \frac{T_{o1}}{T_1}T_1 = \frac{1}{0.9984}278 = 278.4455\mathrm{K}$$

$$T_{o2} = \frac{T_{o2}}{T_2}T_2 = \frac{1}{0.9980}338 = 338.6774\mathrm{K}$$

Solving for the pipe length and inserting the various parameters gives

$$L = \frac{\dot{m}c_p}{\pi D\bar{h}}\ln\frac{T_{o1} - T_s}{T_{o2} - T_s} = \frac{(0.05168)(1004.5)}{\pi(0.05)(140)}\ln\frac{(278.4455 - 433)}{(338.6774 - 433)} = 1.1657\,\mathrm{m}$$

Rayleigh flow pressure drop

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (2.3674)\left(\frac{1}{2.3732}\right)(70) = 69.8289\,\mathrm{kPa}$$

The pressure drop is therefore,

$$(\Delta p)_{Rayleigh} = p_2 - p_1 = 69.8289 - 70 = -0.1711\,\text{kpa}$$

Fanno flow pressure drop

At $M_1 = 0.08976$ from the Fanno relations

$$\left(\frac{p}{p^*}\right)_1 = 12.194$$

$$\left(\frac{fL_{max}}{D}\right)_1 = 83.9637$$

Inserting parameters gives fL/D =(0.018)(1.1657)/0.05 = 0.4197. Therefore,

$$\left(\frac{fL_{max}}{D}\right)_2 = \left(\frac{fL_{max}}{D}\right)_1 - \frac{fL}{D} = 83.9637 - 0.4197 = 83.5440$$

From which we find $M_2 = 0.08998$ and in turn $(p/p^*)_2 = 12.1651$. Thus,

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (12.1651)\left(\frac{1}{12.1943}\right)(70) = 69.8324\,\text{kPa}$$

The pressure drop is therefore,

$$(\Delta p)_{Fanno} = p_2 - p_1 = 69.8324 - 70 = -0.1676\,\text{kpa}$$

The combined pressure drop if added together is

$$(\Delta p) = (\Delta p)_{Rayleigh} + (\Delta p)_{Fanno} = -0.1711 - 0.1676 = -0.3387\,\text{kpa}$$

Problem 9. – Air ($\gamma = 1.4$ and $R = 0.287$ kJ/kg · K) enters a turbojet combustion chamber at 400 K and 200 kPa, with a temperature after combustion of 1000 K. If the heating value of the fuel is 48,000 kJ/kg, determine the required fuel-air ratio (on a mass basis). Assume Rayleigh line flow in the combustion chamber. What fuel-air ratio would be required to choke the combustion chamber? The inlet velocity is 35 m/s.

The air is treated as a perfect gas, so

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)287}{1.4 - 1} = 1004.5\,\mathrm{J/kg \cdot K}$$

$$a_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)(287)(400)} = 400.8990\,\mathrm{m/s}$$

$$M_1 = \frac{V_1}{a_1} = \frac{35}{400.8990} = 0.087304$$

At this Mach number we find from the Rayleigh relations that

$$\left(\frac{T}{T^*}\right)_1 = 0.04298$$
$$\left(\frac{T}{T_o}\right)_1 = 0.9985$$
$$\left(\frac{T_o}{T_o^*}\right)_1 = 0.03587$$

Now

$$\frac{T_2}{T^*} = \frac{T_2}{T_1}\frac{T_1}{T^*} = \frac{1000}{400}(0.04298) = 0.1075$$

From this temperature ratio, we may use the Rayleigh relations to find that $M_2 = 0.14038$. At this Mach number, we find that $(T/T_o)_2 = 0.9961$. T_{o1} and T_{o2} are computed as follows

$$T_{o1} = \frac{T_{o1}}{T_1} T_1 = \frac{1}{0.9985} 400 = 400.6009\mathrm{K}$$

$$T_{o2} = \frac{T_{o2}}{T_2} T_2 = \frac{1}{0.9961} 1000 = 1003.9153\mathrm{K}$$

From an energy balance we have

$$\dot{q} = \dot{m}_a c_p \Delta T_o = \dot{m}_f HV$$

Therefore, the fuel air ratio is

$$\frac{\dot{m}_f}{\dot{m}_a} = \frac{c_p(T_{o2} - T_{o1})}{HV} = \frac{1.0045(1{,}003.9153 - 400.6009)}{48{,}000} = 0.0126$$

To choke the flow $T_{o2} = T_o^*$, where

$$T_o^* = \left(\frac{T_o^*}{T_{o1}}\right) T_{o1} = \left(\frac{1}{0.03587}\right) 400.6009 = 11{,}168.1322\,\text{K}$$

So

$$\frac{\dot{m}_f}{\dot{m}_a} = \frac{c_p\left(T_o^* - T_{o1}\right)}{HV} = \frac{1.0045(11{,}168.1322 - 400.6009)}{48{,}000} = 0.2253$$

Problem 10. - Air ($\gamma = 1.4$, R = 0.287 kJ/kg · K and c_p = 1.0045 kJ/kg · K) flows through a constant-area duct is connected to a reservoir at a temperature of 500ºC and a pressure of 500 kPa by a converging nozzle, as shown in Figure P10.10. Heat is lost at the rate of 250 kJ /kg. (a) Determine the exit exit pressure and Mach number and the mass flow rate for a back pressure of 0 kPa. (b) Determine the exit pressure and Mach number when a normal shock stands in the exit plane of the duct.

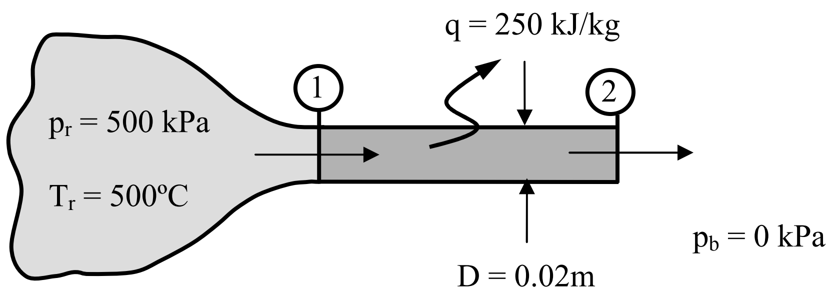


Figure Pl0.10

(a) Because the back pressure is 0 kPa and because heat is removed from the air, the flow in the duct will be supersonic and accelerating. This will occur only if M_1 = 1.0. Therefore, $T_{o1} = T_o^*$ = 773K. From isentropic flow relations, $(T/T_o)_1$ = 0.8333 and $(p/p_o)_1$ = 0.5283. So, T_1 = (0.8333)773= 644.1667K and $p_1 = p^*$ = (0.5283)500 = 264.1500 kPa. From the energy balance on the duct

$$T_{o2} = T_{o1} + \frac{q}{c_p} = 773 - \frac{250}{1.0045} = 524.1200\text{K}$$

$$\frac{T_{o2}}{T_{o1}} = \frac{524.1200}{773} = 0.6780$$

From the Rayleigh relations we find, $M_2 = 2.7613$ and at this Mach number $p_2/p^* = 0.20557$. Accordingly,

$$p_2 = p_e = \left(\frac{p_2}{p^*}\right)p^* = (0.20557)264.1500 = 54.3013\text{kPa}$$

Expansion waves occur outside the duct to allow the pressure to reach the 0 kPA back pressure.

The mass flow rate is computed as follows

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)\left(\frac{\pi}{4}D^2\right)M_1\sqrt{\gamma RT_1}$$

$$= \left[\frac{264.15}{(0.287)(644.1667)}\right]\left[\frac{\pi}{4}(0.02)^2\right]\left[1.0\sqrt{(1.4)(287)(644.1667)}\right] = 0.2284\frac{\text{kg}}{\text{s}}$$

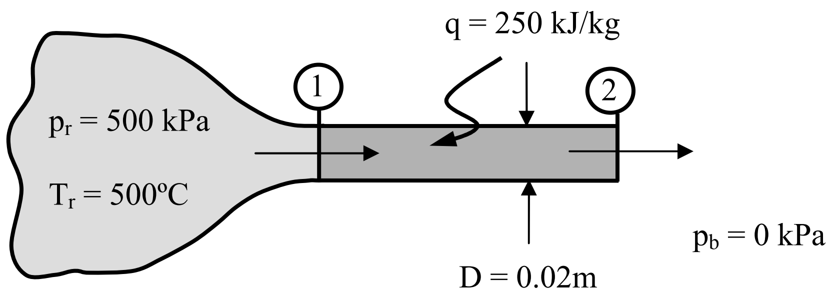
(b) The Mach number just upstream of the shock is 2.7613. From the normal shock relations we find the Mach number on the downstream side to be 0.4910 and the exit pressure is determined by multiplying the static pressure ratio across the shock 8.7289 times the pressure found in part (a), i.e., $p_e = (54.3013)8.7289 = 473.9906\text{kPa} = p_b$.

Problem 11. – Consider flow in a constant-area duct with friction and heat transfer. To maintain a constant subsonic Mach number, should heat be added or removed? Repeat for supersonic flow.

With friction alone, in subsonic flow, M increases, therefore to maintain M constant, remove heat.

With friction alone, in supersonic flow, M decreases, therefore to maintain M constant, remove heat.

Problem 12. – For the system shown in Problem 10, determine the mass flow rate if 250 kJ/kg of heat energy is added to the flow in the duct. The duct diameter is 2 cm. Repeat for a back pressure of 100 kPa. Working fluid is air ($\gamma = 1.4$, $c_p = 1004.5\text{J/ kg} \cdot \text{K}$ and $R = 0.287\ \text{kJ/kg} \cdot \text{K}$).



Assume $M_2 = 1$ so that the duct is choked. Accordingly, $T_{o2} = T_o^*$. Therefore from an energy balance

$$\frac{q}{c_p} = T_{o2} - T_{o1} = T_o^* - T_r$$

$$T_o^* = T_r + \frac{q}{c_p} = 773 + \frac{250}{1.0045} = 1021.8800\text{K}$$

$$\frac{T_{o1}}{T_o^*} = 0.756449$$

At this value, the Rayleigh relations reveal that $M_1 = 0.5473$ and therefore $p_1/p^* = 1.6909$. The isentropic relations at this Mach number provide

$$\frac{p_1}{p_{o1}} = 0.8158 \quad \text{and} \quad \frac{T_1}{T_{o1}} = 0.9435$$

Since $p_{o1} = p_r = 500$ kPa and $T_{o1} = T_r = 773$K, we can use these ratios to compute $p_1 = (0.8158)500 = 407.9000$ kPa and $T_1 = (0.9435)773 = 729.3255$K. Furthermore,

$$p^* = p_e = p_b = \frac{p^*}{p_1} p_1 = \left(\frac{1}{1.6909}\right) 407.9000 = 241.2325\,\text{kPa}$$

Consequently, for $p_b \leq 241.2345$ kPa the duct will be choked due to the heat addition. So for $p_b = 0$ and 100 kPa the maximum flow rate will be realized, which is computed as follows

$$\dot{m} = \rho_1 A_1 V_1 = \left(\frac{p_1}{RT_1}\right)\left(\frac{\pi}{4} D^2\right)\left(M_1 \sqrt{\gamma R T_1}\right)$$

$$= \frac{(407.9)}{(0.287)(729.3255)} \frac{\pi}{4} (0.02)^2 (0.5473)\sqrt{(1.4)(287)(729.3255)}$$

$$= 0.1814 \frac{\text{kg}}{\text{s}}$$

Problem 13. – A detonation wave (Figure P10.13) represents a shock sustained by chemical reaction. Give the continuity, momentum, and energy equations for such a wave, assuming that a chemical reaction taking place in the wave liberates heat q. Denote properties of the unburned gas ahead of the wave by the subscript u and those of the burned gases behind the wave by b. Write the equations for an observer traveling with the wave.

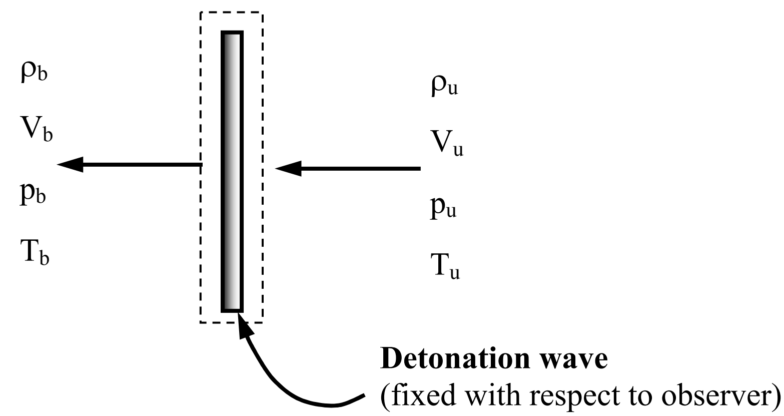


Figure Pl0.l3

Continuity Equation

$$\rho_u V_u = \rho_b V_b$$

Momentum Equation

$$p_u + \rho_u V_u^2 = p_b + \rho_b V_b^2$$

Energy Equation

$$h_u + \frac{V_u^2}{2} + q = h_b + \frac{V_b^2}{2}$$

Problem 14. – Develop a computer program that will yield values of p/p*, T/T*, T_o / T_o^*, and p_o / p_o^* for Rayleigh line flow with the working fluid consisting of a perfect gas with constant γ = 1.36. Use Mach number increments of 0.10 over the range M = 0 to M = 2.5.

The governing relations are

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2} \qquad \textbf{(10.11)}$$

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{\left(1+\gamma M^2\right)^2} \qquad \textbf{(10.12)}$$

$$\frac{T_o}{T_o^*} = \frac{(1+\gamma)M^2[2+(\gamma-1)M^2]}{\left(1+\gamma M^2\right)^2} \qquad \textbf{(10.14)}$$

$$\frac{p_o}{p_o^*} = \left(\frac{1+\gamma M^2}{1+\gamma}\right)\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{\gamma/(\gamma-1)} \qquad \textbf{(10.15)}$$

The following is the spreadsheet computed values

M	p/p*	T/T*	To/To*	po*/po*
0.0	2.36000	0.00000	0.0000	1.2629
0.1	2.29633	0.10750	0.0914	1.2459
0.2	2.18486	0.28137	0.2410	1.2167
0.3	2.03484	0.48651	0.4210	1.1785
0.4	1.86321	0.68061	0.5971	1.1366
0.5	1.68491	0.83637	0.7464	1.0960
0.6	1.51097	0.94328	0.8588	1.0602
0.7	1.34831	1.00299	0.9344	1.0318
0.8	1.20042	1.02352	0.9783	1.0120
0.9	1.06845	1.01467	0.9975	1.0016
1.0	0.95205	0.98562	0.9988	1.0009

1.1	0.85012	0.94381	0.9879	1.0100
1.2	0.76116	0.89483	0.9692	1.0289
1.3	0.68363	0.84266	0.9459	1.0578
1.4	0.61603	0.78995	0.9203	1.0969
1.5	0.55699	0.73843	0.8939	1.1464
1.6	0.50532	0.68913	0.8677	1.2067
1.7	0.45998	0.64262	0.8423	1.2786
1.8	0.42005	0.59918	0.8182	1.3626
1.9	0.38479	0.55886	0.7954	1.4596
2.0	0.35355	0.52160	0.7741	1.5707
2.1	0.32577	0.48727	0.7542	1.6969
2.2	0.30099	0.45569	0.7358	1.8396
2.3	0.27881	0.42666	0.7188	2.0002
2.4	0.25890	0.39999	0.7031	2.1802
2.5	0.24098	0.37547	0.6885	2.3814

Problem 15. – Oxygen (γ = 1.4 and R = 0.2598 kJ/kg · K) is to be pumped through an uninsulated 2.5-cm pipe, 1000 m long (Figure P10.15). A compressor is available at the oxygen source capable of providing a pressure of 1 MPa. If the supply pressure is to be 101 kPa, determine the mass flow rate through the system and the compressor power required. Assume isothermal flow at T = 15˚C.

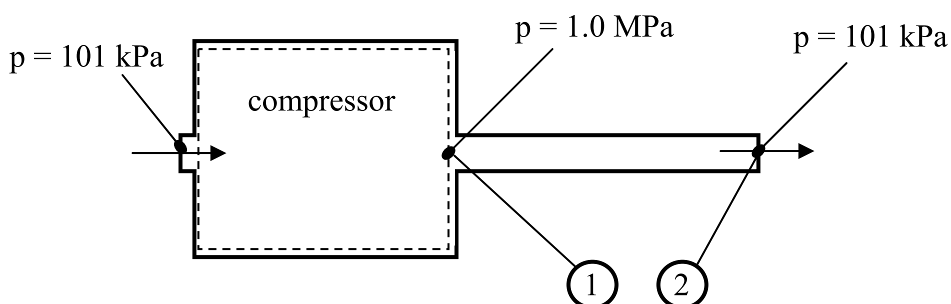


Figure P10.15

Now from Eq.(9.46)

$$\frac{fL}{D} = \frac{1-\gamma M_1^2}{\gamma M_1^2} - \frac{1-\gamma M_2^2}{\gamma M_2^2} + \ln\frac{M_1^2}{M_2^2}$$

The relation between the pressures and the Mach number is contained in Eq.(9.48), i.e.,

$$\frac{p_1}{p_2} = \frac{M_2}{M_1}$$

Calling the ratio p_1/p_2 p and replacing M_2 in the fL/D equation yields after a small amount of algebra

$$\frac{fL}{D} = \left(\frac{1-\gamma M_1^2}{\gamma M_1^2}\right) - \left(\frac{1-\gamma p^2 M_1^2}{\gamma p^2 M_1^2}\right) + \ln\left(\frac{M_1^2}{p^2 M_1^2}\right) = \left(\frac{p^2-1}{\gamma p^2 M_1^2}\right) - \ln\left(p^2\right)$$

Solving for M_1

$$M_1 = \sqrt{\frac{\left(p^2-1\right)}{\gamma p^2\left(\frac{fL}{D} + \ln\left(p^2\right)\right)}}$$

Since $fL/D = (0.018)(1000)/(0.025) = 720$, $(p_1/p_2)^2 = (1000/101)^2 = 98.0296$ and $\gamma = 1.4$, substitution produces $M_1 = 0.0312$. The mass flow rate is computed as follows:

$$\dot{m} = \rho_1 A V_1 = \left(\frac{p_1}{RT_1}\right) A M_1 \sqrt{\gamma R T_1}$$

$$= \left[\frac{(1000)}{(0.2598)288}\right]\left(\frac{\pi}{4}0.025^2\right)0.0312\sqrt{(1.4)(259.8)(288)}$$

$$= 0.0662 \text{ kg/s}$$

For isothermal compression:

$$w_{1-2} = RT\ln\left(\frac{p_2}{p_1}\right) = (0.2598)(288)\ln\left(\frac{101}{1000}\right) = -171.5404 \text{ kJ/kg}$$

The power required is found by multiplying the work by the mass flow rate:

$$P = \dot{m} w_{1-2} = (0.0662)171.5404 = 11.3642 \text{ kW}$$

Problem 16. – Natural gas (assume the properties of methane: $\gamma = 1.32$ and $R = 0.5182$ kJ/kg · K) is to be pumped over a long distance through a 7.5-cm-diameter pipe (Figure P10.16). Assume the gas flow to be isothermal, with T = 15°C. Compressor stations capable of delivering 20 kW to the flow are available, with each compressor capable of raising the gas pressure isothermally to 500 kPa (inlet compressor pressure is to be 120 kPa). How far apart should the compressor stations be located? Assume isothermal compression in each compressor, with f = 0.017.

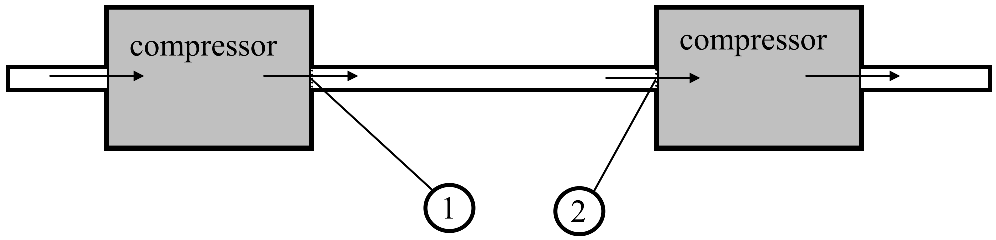


Figure P10.16

For isothermal compression:

$$w_{1-2} = RT\ln\left(\frac{p_2}{p_1}\right) = (0.5182)(288)\ln\left(\frac{120}{500}\right) = -212.9851\ \text{kJ/kg}$$

The power required is equal to the work times the mass flow rate. Therefore, we can determine the mass flow rate as follows:

$$\dot{m} = \frac{P}{w_{1-2}} = \frac{20}{212.9851} = 0.0939\,\text{kg/s}$$

$$V_1 = \frac{\dot{m}}{\rho_1 A} = \frac{RT_1\dot{m}}{p_1 A} = \left[\frac{(0.5182)(288)(0.0939)}{(500)\left(\frac{\pi}{4}0.025^2\right)}\right] = 6.3441\ \text{m/s}$$

Therefore,

$$M_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt{\gamma RT}} = \frac{6.3441}{\sqrt{(1.32)(518.2(288))}} = 0.01429$$

Now from Eq.(9.48)

$$M_2 = \frac{p_1}{p_2}M_1 = \frac{500}{120}(0.01429) = 0.05956$$

From the isothermal relations at M_1 and M_2, we find $(fL_{max}/D)_1 = 3700.6762$ and $(fL_{max}/D)_2 = 207.1945$, thus,

$$\frac{fL}{D} = \left(\frac{fL_{max}}{D}\right)_1 - \left(\frac{fL_{max}}{D}\right)_2 = 3{,}700.6762 - 207.1945 = 3{,}493.4817$$

$$L = (3{,}493.4817)\left(\frac{D}{f}\right) = (3{,}493.4817)\left(\frac{0.075}{0.017}\right) = 15{,}412.4192 \text{ m}$$

Problem 17. – Develop a computer program that will yield values of p/p^*, fL_{max}/D, T_o/T_o^*, and p_o/p_o^* for isothermal flow with the working fluid consisting of a perfect gas with constant $\gamma = 1.34$. Use Mach number increments of 0.10 over the range M = 0.1 to M = 2.5.

The equations that govern this flow are

$$\frac{T_o}{T_o^*} = \left(\frac{\gamma}{3\gamma - 1}\right)\left[2 + (\gamma - 1)M^2\right] \quad \textbf{(10.53)}$$

$$\frac{p}{p^*} = \frac{1}{\sqrt{\gamma}M} \quad \textbf{(10.54)}$$

$$\frac{fL_{max}}{D} = \frac{1 - \gamma M^2}{\gamma M^2} + \ln\left(\gamma M^2\right) \quad \textbf{(10.55)}$$

$$\frac{p_o}{p_o^*} = \frac{1}{\sqrt{\gamma}M}\left\{\frac{\gamma}{(3\gamma - 1)}\left[2 + (\gamma - 1)M^2\right]\right\}^{\frac{\gamma}{\gamma - 1}} \quad \textbf{(10.56)}$$

Spreadsheet computation produces,

M	p/p*	T/T*	p_o*/p_o*	fL_{max}/D
0.100	10.8075	1.1680	5.8600	69.9953
0.200	5.3900	1.1621	2.9817	15.2311
0.300	3.5783	1.1524	2.0461	5.5670
0.400	2.6681	1.1390	1.5974	2.4315
0.500	2.1188	1.1223	1.3446	1.1291
0.600	1.7500	1.1025	1.1912	0.5199
0.700	1.4846	1.0800	1.0961	0.2211

0.800	1.2840	1.0552	1.0390	0.0770
0.900	1.1268	1.0284	1.0091	0.0155
1.000	1.0000	1.0000	1.0000	0.0000
1.100	0.8955	0.9704	1.0082	0.0107
1.200	0.8079	0.9399	1.0314	0.0362
1.300	0.7333	0.9089	1.0687	0.0700
1.400	0.6691	0.8776	1.1195	0.1080
1.500	0.6133	0.8463	1.1839	0.1477
1.600	0.5643	0.8152	1.2624	0.1876
1.700	0.5210	0.7846	1.3557	0.2267
1.800	0.4826	0.7544	1.4650	0.2645
1.900	0.4482	0.7250	1.5913	0.3006
2.000	0.4173	0.6964	1.7364	0.3348
2.100	0.3894	0.6687	1.9020	0.3672
2.200	0.3642	0.6419	2.0902	0.3977
2.300	0.3412	0.6160	2.3031	0.4263
2.400	0.3204	0.5911	2.5434	0.4531
2.500	0.3013	0.5673	2.8139	0.4782

Problem 18. – A subsonic stream of air ($\gamma = 1.4$, $R = 0.287$ kJ/kg · K and $c_p = 1.0045$ kJ/kg · K) flows through a linear, conically shaped, nozzle, i.e., $D = D_i + (D_e - D_i)x/L$. The diameter at the inlet is 2 cm and the diameter at the exit is 5 cm. The nozzle is 10 cm long. The entering Mach number is 0.6. Heat is added to flow at a rate so that the stagnation temperature varies linearly with distance. The stagnation temperature at the inlet is 300K and increases 30K per meter of nozzle. Use Heun's predictor-corrector scheme on a coarse grid that includes 11 grid points to determine the Mach number distribution within the duct. To verify the computations determine the exit Mach number for the case when the heat transfer is zero and compare it to the value determined from isentropic flow computations.

This problem is the same as that described in Example 10.6 with two important exceptions: there is no shock in this problem and the inlet Mach number is subsonic. Accordingly, there is no need to rewrite the governing equation. Only the results of the computations will be shown here.

Exact solution for the adiabatic case

Here we will determine the exit Mach number against which we can contrast the computed value. To obtain this value we follow the usual procedure in which we use the inlet Mach number to find $(A/A^*)_i$ and then determine $(A/A^*)_e$ from the following

$$\frac{A_e}{A^*} = \frac{A_e}{A_i}\frac{A_i}{A^*} = \left(\frac{D_e}{D_i}\right)^2 \frac{A_i}{A^*}$$

Now at Mi = 0.6, we find $(A/A^*)_i$ 1.1882 and for De = 0.05m and Di = 0.02m, we obtain $(A/A^*)_e = 7.4263$. At this value we find Me = 0.07821.

Numerical solution for the adiabatic case

Here the grid is divided into 10 pieces, i.e., $\Delta x = 0.01$m and the results from applying Heun's method are contained in the following table

pt	x	M_i	$F(x_i,M_i)$	M_p	$F(x_{i+1},M_p)$	M_{i+1}
1	0.0000	0.6000	-30.1500	0.2985	-8.7010	0.4057
2	0.0100	0.4057	-13.0878	0.2749	-6.9652	0.3055
3	0.0200	0.3055	-7.9202	0.2263	-4.9848	0.2410
4	0.0300	0.2410	-5.3540	0.1874	-3.6675	0.1958
5	0.0400	0.1958	-3.8479	0.1574	-2.7800	0.1627
6	0.0500	0.1627	-2.8803	0.1339	-2.1606	0.1375
7	0.0600	0.1375	-2.2213	0.1153	-1.7144	0.1178
8	0.0700	0.1178	-1.7534	0.1003	-1.3843	0.1021
9	0.0800	0.1021	-1.4104	0.0880	-1.1344	0.0894
10	0.0900	0.0894	-1.1525	0.0779	-0.9415	0.0789
11	0.1000	0.07894				

Thus, the accuracy is (0.07894/0.07821 – 1)100 = 0.93%. However, using a grid containing 161 grid points produces $M_e = 0.07822$, which differs by only 0.012%.

Now introducing heat transfer into the computations on a coarse grid of 11 grid points yields

pt	x	M_i	$F(x_i,M_i)$	M_p	$F(x_{i+1},M_p)$	M_{i+1}
1	0.0000	0.6000	-30.0744	0.2993	-8.7093	0.4061
2	0.0100	0.4061	-13.0727	0.2754	-6.9632	0.3059
3	0.0200	0.3059	-7.9144	0.2268	-4.9838	0.2414
4	0.0300	0.2414	-5.3516	0.1879	-3.6675	0.1963
5	0.0400	0.1963	-3.8472	0.1578	-2.7805	0.1632
6	0.0500	0.1632	-2.8804	0.1344	-2.1615	0.1380
7	0.0600	0.1380	-2.2220	0.1157	-1.7155	0.1183
8	0.0700	0.1183	-1.7543	0.1007	-1.3855	0.1026
9	0.0800	0.1026	-1.4115	0.0885	-1.1356	0.0898
10	0.0900	0.0898	-1.1536	0.0783	-0.9427	0.0794
11	0.1000	0.07936				

This exit Mach number differs from the adiabatic value by only (0.07936/0.07894 – 1)100 = 0.53%. Therefore, heat transfer in this flow is not important.

Problem 19. – A supersonic stream of air (γ = 1.4, R = 0.287 kJ/kg · K and c_p = 1.004 kJ/kg · K) flows through a linear, conically shaped, nozzle, i.e., $D = D_i + (D_e - D_i)x/L$. The diameter at the inlet is 2 cm and the diameter at the exit is 5 cm. The nozzle is 10 cm long. The entering Mach number is 3. Heat is added to flow at a rate so that the stagnation temperature varies linearly with distance. The stagnation temperature at the inlet is 300K and increases 30K per meter of nozzle. The pressure is such that a normal shock wave stands half way down the nozzle. Use Euler's explicit method to determine the Mach number distribution within the duct.

This problem is the same as Example 10.6 except that it uses Euler's explicit method instead of Heun's. As a demonstration of the expected accuracy, the adiabatic problem with no shock is solved on a variety of grids and the results are compared to the value obtained using isentropic area relations, i.e., 5.071544. The results are also compared to the results obtained by using Heun's method.

Euler's Adiabatic Results

Pts	Δx	M (x=10)	% error
11	0.01	5.167509	1.8922
21	0.005	5.118790	0.9316
41	0.0025	5.094975	0.4620
81	0.00125	5.083206	0.2300
161	0.00625	5.077357	0.1146

Heun's Adiabatic Results

Pts	Δx	M (x=10)	% error
11	0.01	5.076171	0.0912
21	0.005	5.072687	0.0225
41	0.0025	5.071820	0.0054
81	0.00125	5.071603	0.0012
161	0.00625	5.071550	0.0001

As can be seen, Heun's method, which is a 2nd order method produces more accuracy for the same grid size. Euler's results are not great particularly at larger grid sizes. Nonetheless, the following are summary results for the problem using Euler on the smallest grid in the table above.

				Euler	Heun	
pt	**x**	**M_i**	**$F(x_i,M_i)$**	**M_{i+1}**	**M_{i+1}**	**% diff**
1	0.0000	3.0000	31.5000	3.0197	3.0191	0.02
17	0.0100	3.2958	27.6569	3.3131	3.3034	0.29
33	0.0200	3.5587	24.8735	3.5742	3.5541	0.57
49	0.0300	3.7970	22.7378	3.8112	3.7792	0.85
65	0.0400	4.0161	21.0331	4.0292	3.9837	1.14
81	0.0500	4.21964		0.4295	0.4309	-0.33
82	0.0500	0.4295	-9.3608	0.4236	0.4251	-0.35
98	0.0600	0.3509	-6.4740	0.3469	0.3490	-0.61
114	0.0700	0.2946	-4.8037	0.2916	0.2940	-0.80
130	0.0800	0.2520	-3.7168	0.2497	0.2521	-0.95
146	0.0900	0.2186	-2.9593	0.2168	0.2192	-1.09
162	0.1000	0.1918			0.1941	-1.21

Problem 20. – A supersonic stream of air ($\gamma = 1.3$, R = 0.287 kJ/kg · K and $c_p = 1.004$ kJ/kg · K) flows through a linear, conically shaped, nozzle, i.e., $D = D_i + (D_e - D_i)x/L$. The diameter at the inlet is 2 cm and the diameter at the exit is 5 cm. The nozzle is 10 cm long. The entering Mach number is 3. Heat is added to flow at a rate so that the stagnation temperature varies linearly with distance. The stagnation temperature at the inlet is 300K and increases 30K per meter of nozzle. The pressure is such that a normal shock wave stands half way down the nozzle. Use Heun's predictor-corrector method to determine the Mach number distribution within the duct.

This problem is the same as Example 10.6 except that it uses $\gamma = 1.3$ instead of $\gamma = 1.4$. The computed results follow

pt	x	M_i	$F(x_i,M_i)$	M_p	$F(x_{i+1},M_p)$	M_{i+1}
1	0.0000	3.0000	25.8779	3.0259	25.4577	3.0257
17	0.0100	3.2397	22.2699	3.2536	22.0810	3.2536
33	0.0200	3.4485	19.6154	3.4608	19.4719	3.4608
49	0.0300	3.6340	17.5579	3.6450	17.4442	3.6450
65	0.0400	3.8010	15.9031	3.8110	15.8101	3.8110
81	0.0500	3.95303				0.4072
82	0.0500	0.4072	-8.5469	0.4019	-8.3413	0.4020
98	0.0600	0.3354	-6.0469	0.3317	-5.9303	0.3317
114	0.0700	0.2831	-4.5404	0.2802	-4.4655	0.2803
130	0.0800	0.2430	-3.5386	0.2408	-3.4869	0.2408
146	0.0900	0.2113	-2.8311	0.2096	-2.7938	0.2096
162	0.1000	0.1857				

Chapter Eleven

EQUATIONS OF MOTION FOR MULTIDIMENSIONAL FLOW

Problem 1. – Prove that for a perfect gas

(a) $p=\rho(\gamma-1)e$

(b) $e_t=\dfrac{a^2}{\gamma(\gamma+1)}+\dfrac{V^2}{2}$

(a) For a perfect gas

$$p=\rho RT \tag{1}$$

$$e=c_v T \tag{2}$$

$$c_v=\frac{R}{\gamma-1} \tag{3}$$

Combine Eqs. (2) and (3) and rearrange to get

$$RT=e(\gamma-1) \tag{4}$$

Substitute Eq. (4) into Eq. (1) to the result for part (a)

$$p=\rho(\gamma-1)e$$

(b) The definition of the total (internal) energy is

$$e_t=e+\frac{V^2}{2}$$

Combining this with Eq. (4) brings

$$e_t=\frac{RT}{\gamma-1}+\frac{V^2}{2}$$

Substitute $a^2=\gamma RT$ in the above equation to obtain the result for part (b)

$$e_t=\frac{a^2}{\gamma(\gamma+1)}+\frac{V^2}{2}$$

Problem 2. – According to the generalized continuity equation given by Eq. (11.1), for steady, incompressible, one-dimensional flow, $\partial u/\partial x = 0$, or, in other words, u is equal to a constant. Previously, for incompressible flow, however, it has been customary to assume that, for steady, one-dimensional flow, the product of velocity and cross-sectional area (AV) is a constant. Explain this seeming contradiction.

Quasi one-dimensional flow is a one-dimensional approximation to a class of flows that are three-dimensional in actuality. This approximation is realized by assuming a uniform axial velocity in each cross section. Equation (11.45) describes the original three-dimensional flow. The approximation, however, cannot be made by simply letting v and w vanish in Eq.(11.45), because this would obviously cause a mass imbalance. The eliminated v and w velocity components should be compensated for by bringing about changes in u, at different cross sections. This is performed by integrating Eq.(1.21) over a control volume that is constrained between two cross sections of interest, and this leads to VA = constant, as explained in previous chapters.

Problem 3. – The continuity equation for steady two-dimensional flow is

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

A function ψ (the compressible *stream function*) may be defined so that this equation is automatically satisfied. Show that the following accomplish this

$$u = -\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial x}$$

where ρ_∞ is a constant that is inserted so that the stream function has the same units as the incompressible flow stream function. What are the units of the stream function? What are the units of the velocity potential, ϕ?

Differentiation of the above equations yield

$$\frac{\partial(\rho u)}{\partial x} = -\rho_\infty \frac{\partial^2 \psi}{\partial y \partial x}$$

$$\frac{\partial(\rho v)}{\partial y} = \rho_\infty \frac{\partial^2 \psi}{\partial x \partial y}$$

Therefore

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=\rho_\infty\left(\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x}\right)$$

But for a continuous stream function with continuous derivatives

$$\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial^2\psi}{\partial y\partial x}$$

and, hence, the continuity equation is automatically satisfied.

The dimensions of both the stream function and velocity potential are: L^2T^{-1}.

Problem 4. – Expand Eq.(11.10) into the three component equations, and show that Eqs.(11.7), (11.8), and (11.9) result.

Equation (11.10) can be written as

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}\right)=\frac{\partial u}{\partial t}\mathbf{i}+\frac{\partial v}{\partial t}\mathbf{j}+\frac{\partial w}{\partial t}\mathbf{k}+$$
$$\left[(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\bullet\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\right](u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$$

or

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}\right)=\frac{\partial u}{\partial t}\mathbf{i}+\frac{\partial v}{\partial t}\mathbf{j}+\frac{\partial w}{\partial t}\mathbf{k}+$$
$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)\mathbf{i}+\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\mathbf{j}+\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)\mathbf{k}$$

This implies that

$$\begin{cases}-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}=\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}\\ -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}=\dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}\\ -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}=\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}\end{cases}$$

Problem 5. – Show that

$$\frac{dp}{\rho}+d\left(\frac{V^2}{2}\right)=0 \tag{1}$$

and

$$\frac{\nabla p}{\rho}+\nabla\left(\frac{V^2}{2}\right)=0 \tag{2}$$

are equivalent.

Take the dot product of Eq. (2) with the differential displacement vector, $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ to get

$$\left[\frac{\nabla p}{\rho}+\nabla\left(\frac{V^2}{2}\right)\right]\bullet dr=0$$

or

$$\frac{1}{\rho}(\nabla p \bullet d\mathbf{r})+\nabla\left(\frac{V^2}{2}\right)\bullet d\mathbf{r}=0$$

which can also be written as

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}\right)\bullet(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})+$$
$$\left[\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)\mathbf{i}+\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)\mathbf{j}+\frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)\mathbf{k}\right]\bullet(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})=0$$

or

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right)+\left[\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz\right]=0 \tag{3}$$

Since Eqs. (1) and (2), have been derived for steady flow, it follows that

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz$$

and

$$d\left(\frac{V^2}{2}\right)=\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)dx+\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right)dy+\frac{\partial}{\partial z}\left(\frac{V^2}{2}\right)dz$$

Substitution of these into Eq.(3) produces Eq. (1).

Problem 6. – Derive Eq.(11.8), i.e., prove that

$$\rho\frac{Dh}{Dt}=\rho\dot{q}+\frac{Dp}{Dt}$$

Consider the continuity equation

$$\frac{D\rho}{Dt}+\rho\nabla\bullet\mathbf{V}=0$$

or

$$\nabla\bullet\mathbf{V}=-\frac{1}{\rho}\frac{D\rho}{Dt} \tag{1}$$

Also since $h=e+\frac{p}{\rho}$ or $e=h-\frac{p}{\rho}$, then differentiation yields

$$\frac{De}{Dt}=\frac{Dh}{Dt}-\frac{D}{Dt}\left(\frac{p}{\rho}\right)$$

or

$$\frac{De}{Dt}=\frac{Dh}{Dt}-\frac{\rho\,Dp/Dt-p\,D\rho/Dt}{\rho^2}$$

or

$$\rho\frac{De}{Dt}=\rho\frac{Dh}{Dt}-\frac{Dp}{Dt}+\frac{p}{\rho}\frac{D\rho}{Dt} \tag{2}$$

Substitution of Eqs.(1) and (2) in Eq.(11.17), then leads to

$$\rho\frac{Dh}{Dt}-\frac{Dp}{Dt}+\frac{p}{\rho}\frac{D\rho}{Dt}=\rho\dot{q}+\frac{p}{\rho}\frac{D\rho}{Dt}$$

which simplifies to give Eq.(11.18).

Problem 7. – Under what conditions can it be assumed that $p + \rho \frac{V^2}{2}$ is equal to a constant?

Consider the dot product $\mathbf{V} \bullet \nabla \left(\frac{p}{\rho} + \frac{V^2}{2} \right)$. It can be expanded as follows

$$\mathbf{V} \bullet \nabla \left(\frac{p}{\rho} + \frac{V^2}{2} \right) = \mathbf{V} \bullet \nabla \left(\frac{p}{\rho} \right) + \mathbf{V} \bullet \nabla \left(\frac{V^2}{2} \right) = \mathbf{V} \bullet \left(\frac{\rho \nabla p - p \nabla \rho}{\rho^2} \right) + \mathbf{V} \bullet \nabla \left(\frac{V^2}{2} \right)$$

$$= \mathbf{V} \bullet \left[\frac{\nabla p}{\rho} + \nabla \left(\frac{V^2}{2} \right) \right] - \mathbf{V} \bullet \frac{p \nabla \rho}{\rho^2}$$

But using the vector form of Eq.(11.34)

$$\frac{\nabla p}{\rho} + \nabla \left(\frac{V^2}{2} \right) = 0$$

for irrotational, steady, frictionless flow with no external forces except pressure. Therefore the dot product introduced above becomes

$$\mathbf{V} \bullet \nabla \left(\frac{p}{\rho} + \frac{V^2}{2} \right) = -\mathbf{V} \bullet \frac{p \nabla \rho}{\rho^2}$$

This means that, if the right hand side term is zero for a flow, then $\nabla \left(\frac{p}{\rho} + \frac{V^2}{2} \right) \perp \mathbf{V}$, in other words, $\frac{p}{\rho} + \frac{V^2}{2} = \text{Constant}$ along a streamline. Now, in order for the term $\mathbf{V} \bullet \frac{p \nabla \rho}{\rho^2}$ to vanish, we must have either $\nabla \rho = 0$ or $\nabla \rho \perp \mathbf{V}$. The former is a constant density flow (for our steady assumption), and we can show that the latter also leads to a constant density flow:

The continuity equation

$$\nabla \bullet (\rho \mathbf{V}) = 0$$

can be expanded as

$$\rho \nabla \bullet \mathbf{V} + \mathbf{V} \bullet \nabla \rho = 0$$

Using this, the assumption $\nabla\rho \perp \mathbf{V}$, which implies $\mathbf{V} \bullet \nabla\rho = 0$, leads to $\nabla \bullet \mathbf{V} = 0$. This is again the continuity description of a constant density flow.

Problem 8. – Show that Crocco's equation along a streamline can be written as

$$\frac{1}{2}\frac{\partial V^2}{\partial t} = T\,\mathbf{V} \bullet \nabla s - \mathbf{V} \bullet \nabla h_o .$$

Now

$$\mathbf{V} \bullet \frac{\partial \mathbf{V}}{\partial t} = (u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k}) \bullet \left(\frac{\partial u}{\partial t}\mathbf{i} + \frac{\partial v}{\partial t}\mathbf{j} + \frac{\partial w}{\partial t}\mathbf{k}\right)$$

$$= u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + w\frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(u^2 + v^2 + w^2\right) = \frac{1}{2}\frac{\partial V^2}{\partial t}$$

Take the dot product of Eq.(11.39) with the velocity vector to get

$$T\,\mathbf{V} \bullet \nabla s - \mathbf{V} \bullet \nabla h_o = \mathbf{V} \bullet \frac{\partial \mathbf{V}}{\partial t} - 2\,\mathbf{V} \bullet (\mathbf{V} \times \boldsymbol{\omega})$$

By definition, the product $\mathbf{V} \times \boldsymbol{\omega}$ is perpendicular to both vectors $\mathbf{V}$ and $\boldsymbol{\omega}$. Therefore, $\mathbf{V} \bullet (\mathbf{V} \times \boldsymbol{\omega}) = 0$, and the above equation becomes

$$T\,\mathbf{V} \bullet \nabla s - \mathbf{V} \bullet \nabla h_o = \mathbf{V} \bullet \frac{\partial \mathbf{V}}{\partial t}$$

Combining the two previous expressions produces

$$\frac{1}{2}\frac{\partial V^2}{\partial t} = T\,\mathbf{V} \bullet \nabla s - \mathbf{V} \bullet \nabla h_o$$

Problem 9. – Using the substantial derivative operator within Crocco's equation, Eq.(11.39), develop the following equation for the entropy

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \mathbf{V} \bullet \nabla s = \frac{\partial s}{\partial t} + \frac{1}{T}\mathbf{V} \bullet \left(\nabla h_o + \frac{\partial \mathbf{V}}{\partial t}\right)$$

Under what conditions will the entropy remain constant along a streamline?

As in problem 8, take the dot product of Eq.(11.39) with the velocity vector to get

$$T\,\mathbf{V}\bullet\nabla s-\mathbf{V}\bullet\nabla h_o=\mathbf{V}\bullet\frac{\partial\mathbf{V}}{\partial t}-2\,\mathbf{V}\bullet(\mathbf{V}\times\boldsymbol{\omega})$$

By definition, the product $\mathbf{V}\times\boldsymbol{\omega}$ is perpendicular to both vectors $\mathbf{V}$ and $\boldsymbol{\omega}$. Therefore, $\mathbf{V}\bullet(\mathbf{V}\times\boldsymbol{\omega})=0$, and the above equation becomes

$$T\,\mathbf{V}\bullet\nabla s-\mathbf{V}\bullet\nabla h_o=\mathbf{V}\bullet\frac{\partial\mathbf{V}}{\partial t}\tag{1}$$

or

$$\mathbf{V}\bullet\nabla s=\frac{1}{T}\mathbf{V}\bullet\left(\nabla h_o+\frac{\partial\mathbf{V}}{\partial t}\right)\tag{2}$$

Now add $\partial s/\partial t$ to both sides of Eq.(2) to get

$$\frac{Ds}{Dt}=\frac{\partial s}{\partial t}+\mathbf{V}\bullet\nabla s=\frac{\partial s}{\partial t}+\frac{1}{T}\mathbf{V}\bullet\left(\nabla h_o+\frac{\partial\mathbf{V}}{\partial t}\right)\tag{3}$$

Now consider Eq.(2) again. For a steady adiabatic flow $\mathbf{V}\bullet\nabla h_o=0$ and $\dfrac{\partial\mathbf{V}}{\partial t}=0$. Therefore, for a steady adiabatic flow, the right hand side of Eq.(2) vanishes, i.e.,

$$\mathbf{V}\bullet\nabla s=0$$

Put this into Eq.(3), along with the steady flow assumption, to obtain

$$\frac{Ds}{Dt}=\frac{\partial s}{\partial t}+\mathbf{V}\bullet\nabla s=0$$

which indicates that for steady, adiabatic flow, entropy is constant along a streamline.

Problem 10. – From vector mechanics it is known that the curl of a gradient ($\nabla\times\nabla f$) is identically zero. Demonstrate that this is true and use this face to prove that if a velocity potential ($V=\nabla\phi$) exists the flow is irrotational.

By definition

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \right)\mathbf{k} = 0$$

The rotation vector is defined as $\boldsymbol{\omega} = \frac{1}{2}(\nabla \times \mathbf{V})$. If $\mathbf{V} = \nabla\phi$, the rotation vector becomes

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla \times \nabla\phi) = 0$$

The flow is, therefore, irrotational.

Problem 11. – The velocity components for a possible flow field are given by $u = -3x^2 + 2y$ and $v = 2x + 2y$. Is the flow irrotational? If so, determine the velocity potential.

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla \times \mathbf{V}) = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k} \right]$$

Since the flow if two-dimensional, the above relation simplifies to

$$\boldsymbol{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\mathbf{k}$$

But $\frac{\partial u}{\partial y} = 2$ and $\frac{\partial v}{\partial x} = 2$. The rotation vector then becomes

$$\boldsymbol{\omega} = \frac{1}{2}(2 - 2)\mathbf{k} = 0$$

Therefore, the flow is irrotational.

Based on definition of the velocity potential, Eq.(11.4), $u = \dfrac{\partial \phi}{\partial x}$ and $v = \dfrac{\partial \phi}{\partial y}$. Therefore,

$$\phi = \int u\,dx + F(y) = \int \left(-3x^2 + 2y\right) dx + F(y)$$

or

$$\phi = -x^3 + 2yx + F(y) \tag{1}$$

Differentiation with respect to y produces

$$\frac{\partial \phi}{\partial y} = 2x + \frac{dF}{dy} \tag{2}$$

But

$$\frac{\partial \phi}{\partial y} = v = 2x + 2y$$

Substitute this into Eq.(2) to get

$$\frac{dF}{dy} = 2y$$

Integration then gives

$$F(y) = y^2 + c \tag{3}$$

where c is a constant. Finally, substitute Eq.(3) into Eq.(1) to obtain

$$\phi = -x^3 + 2yx + y^2 + c$$

Problem 12. – Show that the velocity potential equation, Eq.(11.47) can be written in two-dimensional form as

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial^2 \phi}{\partial x^2} - 2\frac{uv}{a^2}\frac{\partial^2 \phi}{\partial x \partial y} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial^2 \phi}{\partial y^2} = 0$$

Equation (11.47), in two dimensions, can be written as

$$a^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) - \left[\left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial x^2} + \left(\frac{\partial \phi}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial y^2}\right] - 2\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

Rearrangement leads to

$$\left[a^2-\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2}+\left[a^2-\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2}-2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}=0$$

Divide the above equation by a^2 and use the definition of the velocity potential to get

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\phi}{\partial x^2}-2\frac{u\,v}{a^2}\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\phi}{\partial y^2}=0$$

Problem 13. – Consider a steady, uniform flow of air (γ = 1.4, R = 0.287 kJ/kg.K) with velocity components u = 120 m/s and v = w = 0. Determine the velocity potential, substitute into Eq.(11.48), and find the resultant difference between static and stagnation temperature.

Using the definition of the velocity potential and the above velocity components, we get

$$\frac{\partial\phi}{\partial x}=120$$

Since the other velocity components are zero, the velocity potential is a function of x only. Therefore

$$\phi=120\ x+c$$

and

$$\nabla\phi=120\ \mathbf{i}$$

Since

$$a_o^2=a^2+\frac{\gamma-1}{2}\nabla\phi\bullet\nabla\phi$$

we get

$$a_o^2-a^2=\gamma R\,(T_o-T)=\frac{\gamma-1}{2}\nabla\phi\bullet\nabla\phi$$

Therefore

$$T_o - T = \frac{\gamma - 1}{2\gamma R}\nabla\phi \bullet \nabla\phi = \frac{(1.4-1)}{(2)(1.4)(287)}(120)^2 \cong 7.1677\,\mathrm{K}$$

Problem 14. – Using the stream function, as defined in Problem 11.3, develop the following expression for steady, two-dimensional, irrotational flow

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial^2 \psi}{\partial x^2} - 2\frac{uv}{a^2}\frac{\partial^2 \psi}{\partial x \partial y} + \left(1 - \frac{v^2}{a^2}\right)\frac{\partial^2 \psi}{\partial y^2} = 0$$

From irrotational flow

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

and from the definition of the stream function

$$u = \frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial y}$$

$$v = -\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial x}$$

Hence,

$$\frac{\partial}{\partial x}\left(-\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\rho_\infty}{\rho}\frac{\partial \psi}{\partial y}\right) = 0$$

$\therefore$

$$\frac{\rho_\infty}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\rho_\infty}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial \psi}{\partial y} - \frac{\rho_\infty}{\rho^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0$$

$\therefore$

$$\rho_\infty\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \frac{\rho_\infty}{\rho}\frac{\partial \rho}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\rho_\infty}{\rho}\frac{\partial \rho}{\partial y}\frac{\partial \psi}{\partial y}$$

Now for isentropic flow

$$\frac{\partial p}{\partial x} = a^2\frac{\partial \rho}{\partial x}$$

$$\frac{\partial p}{\partial y} = a^2\frac{\partial \rho}{\partial y}$$

$$\frac{\rho_\infty}{\rho}\frac{\partial\rho}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\rho_\infty}{\rho}\frac{\partial\rho}{\partial y}\frac{\partial\psi}{\partial y}=-\frac{\partial p}{\partial x}\frac{v}{a^2}+\frac{\partial p}{\partial y}\frac{u}{a^2}$$

From Euler's equations

$$v\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{v}{\rho}\frac{\partial p}{\partial x}$$

$$-u\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{u}{\rho}\frac{\partial p}{\partial y}$$

So

$$-u^2\frac{\partial v}{\partial x}+v^2\frac{\partial u}{\partial y}+uv\frac{\partial u}{\partial x}-uv\frac{\partial v}{\partial y}=\frac{a^2}{\rho}\left(-\frac{\partial p}{\partial x}\frac{v}{a^2}+\frac{\partial p}{\partial y}\frac{u}{a^2}\right)$$

Thus,

$$-\frac{\partial p}{\partial x}\frac{v}{a^2}+\frac{\partial p}{\partial y}\frac{u}{a^2}=\frac{1}{a^2}\left(-\rho u^2\frac{\partial v}{\partial x}+\rho v^2\frac{\partial u}{\partial y}+\rho uv\frac{\partial u}{\partial x}-\rho uv\frac{\partial v}{\partial y}\right)$$

$$=-\frac{u^2}{a^2}\left[\frac{\partial(\rho v)}{\partial x}-v\frac{\partial\rho}{\partial x}\right]+\frac{v^2}{a^2}\left[\frac{\partial(\rho u)}{\partial y}-u\frac{\partial\rho}{\partial y}\right]$$
$$+\frac{uv}{a^2}\left[\frac{\partial(\rho u)}{\partial x}-u\frac{\partial\rho}{\partial x}\right]-\frac{uv}{a^2}\left[\frac{\partial(\rho v)}{\partial y}-v\frac{\partial\rho}{\partial y}\right]$$

$$-\frac{\partial p}{\partial x}\frac{v}{a^2}+\frac{\partial p}{\partial y}\frac{u}{a^2}=-\frac{u^2}{a^2}\frac{\partial(\rho v)}{\partial x}+\frac{v^2}{a^2}\frac{\partial(\rho u)}{\partial y}+\frac{uv}{a^2}\left[\frac{\partial(\rho u)}{\partial x}-\frac{\partial(\rho v)}{\partial y}\right]$$

$$=\rho_\infty\frac{u^2}{a^2}\frac{\partial^2\psi}{\partial x^2}+\rho_\infty\frac{v^2}{a^2}\frac{\partial^2\psi}{\partial y^2}+\rho_\infty\frac{uv}{a^2}\left(\frac{\partial^2\psi}{\partial x\partial y}+\frac{\partial^2\psi}{\partial x\partial y}\right)$$

$$=\rho_\infty\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$

Rearranging produces the result

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\psi}{\partial x^2}-\frac{2uv}{a^2}\frac{\partial^2\psi}{\partial x\partial y}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\psi}{\partial y^2}=0$$

Problem 15. – Use the technique presented in Example 11.4 to write (a) the velocity components in terms of the velocity potential in spherical coordinates and (b) the steady energy equation for three-dimensional, adiabatic flow in cylindrical coordinates.

a)

$$\begin{aligned} u_1 &= r & h_1 &= 1 \\ u_2 &= \theta & h_2 &= r \\ u_3 &= \varphi & h_3 &= r\sin\theta \end{aligned}$$

$$\mathbf{V} = (V_1, V_2, V_3) = \nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\mathbf{e_1} + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\mathbf{e_2} + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\mathbf{e_3}$$

$$V_1 = V_r = \frac{\partial\phi}{\partial r}$$

$$V_2 = V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$$V_3 = V_\varphi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}$$

b) From Eq.(11.14)

$$\rho\mathbf{V} \bullet \nabla e_t + \nabla \bullet (p\mathbf{V}) = 0$$

$$\begin{aligned} V_1 &= V_r & h_1 &= 1 \\ V_2 &= V_\theta & h_2 &= r \\ V_3 &= V_z & h_3 &= 1 \end{aligned}$$

$$\rho\mathbf{V} \bullet \nabla e_t + \nabla \bullet (p\mathbf{V}) = 0$$

$$\rho V_r \frac{\partial e_t}{\partial r} + \frac{\rho V_\theta}{r}\frac{\partial e_t}{\partial\theta} + \rho V_z \frac{\partial e_t}{\partial z} + \frac{1}{r}\left[\frac{\partial}{\partial r}(prV_r) + \frac{\partial}{\partial\theta}(pV_\theta) + \frac{\partial}{\partial z}(prV_z)\right] = 0$$

Chapter Twelve

EXACT SOLUTIONS

Problem 1. – The velocity potential equation can be written in a variety of forms. For example, Taylor and Maccoll, Ref. 23, used the following forms for problems in Cartesian coordinates

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{a^2}\left[\left(\frac{\partial \varphi}{\partial x}\right)\frac{\partial V^2}{\partial x} + \left(\frac{\partial \varphi}{\partial y}\right)\frac{\partial V^2}{\partial y}\right] \tag{1}$$

Derive this expression and then show that it may be written as

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{a^2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2\frac{\partial^2 \varphi}{\partial x^2} + 2\left(\frac{\partial \varphi}{\partial x}\right)\left(\frac{\partial \varphi}{\partial y}\right)\frac{\partial^2 \varphi}{\partial x \partial y} + \left(\frac{\partial \varphi}{\partial y}\right)^2\frac{\partial^2 \varphi}{\partial y^2}\right] \tag{2}$$

Equation (11.47), in two dimensions, can be written as

$$a^2\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) - \left[\left(\frac{\partial \varphi}{\partial x}\right)^2\frac{\partial^2 \varphi}{\partial x^2} + \left(\frac{\partial \varphi}{\partial y}\right)^2\frac{\partial^2 \varphi}{\partial y^2}\right] - 2\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} = 0$$

or

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{a^2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2\frac{\partial^2 \varphi}{\partial x^2} + \left(\frac{\partial \varphi}{\partial y}\right)^2\frac{\partial^2 \varphi}{\partial y^2} + 2\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y}\right] \tag{3}$$

Note Eqs. (2) and (3) are identical. Further

$$\frac{\partial V^2}{\partial x} = \frac{\partial}{\partial x}\left(u^2 + v^2\right) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

and

$$\frac{\partial V^2}{\partial y} = \frac{\partial}{\partial y}\left(u^2 + v^2\right) = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$$

therefore

$$u\frac{\partial V^2}{\partial x} + v\frac{\partial V^2}{\partial y} = 2u^2\frac{\partial u}{\partial x} + 2uv\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2v^2\frac{\partial v}{\partial y}$$

which leads to

$$u^2\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial y}=\frac{1}{2}\left(u\frac{\partial V^2}{\partial x}+v\frac{\partial V^2}{\partial y}\right)-u\,v\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

Using the definition of velocity potential, the above equation becomes

$$\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}+\left(\frac{\partial\varphi}{\partial y}\right)^2\frac{\partial^2\varphi}{\partial y^2}=\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\frac{\partial V^2}{\partial x}+\frac{\partial\varphi}{\partial y}\frac{\partial V^2}{\partial y}\right)-\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\left(2\frac{\partial^2\varphi}{\partial x\partial y}\right)$$

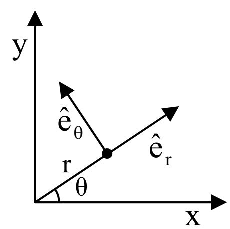
Substitution of this into the right hand side of the Eq. (3) then yields Eq. (1).

Problem 2. – Show that polar velocity components v_r and v_θ are related to the Cartesian velocity components u and v by

$$v_r = u\cos\theta + v\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta$$

Find the inverse of these, i.e., develop expressions for $u = u\,(v_r, v_\theta)$ and $v = v\,(v_r, v_\theta)$.

Consider the following figure



where $\hat{e}_r$ and $\hat{e}_\theta$ are the unit normal vectors of the polar coordinate system. These unit vectors are related to the Cartesian unit vectors, $\hat{i}$ and $\hat{j}$, through a counterclockwise rotation of angle θ. In other words

$$\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\end{bmatrix} \tag{1}$$

Now consider the velocity vector expressed in both coordinate systems

$$\vec{V} = v_r\,\hat{e}_r + v_\theta\,\hat{e}_\theta = u\,\hat{i} + v\,j \tag{2}$$

Substituting for the polar unit vectors from Eq. (1) in the above yields

$$v_r\left(\cos\theta\,\hat{i}+\sin\theta\,\hat{j}\right)+v_\theta\left(-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}\right)=u\,\hat{i}+v\,\hat{j}$$

which after equating components gives

$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

Taking the inverse of Eq. (1) leads to

$$\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix}$$

Using this and substituting for the Cartesian unit vectors in Eq.(2)

$$v_r \hat{e}_r + v_\theta \hat{e}_\theta = u(\cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta) + v(\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta)$$

which after equating components gives

$$v_r = u\cos\theta + v\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta$$

Problem 3. – For a uniform flow in the +y direction, show that the expressions for the velocity components in the previous problem give

$$v = V, \quad u = 0, \quad v_r = V\sin\theta \quad \text{and} \quad v_\theta = V\cos\theta$$

Obtain expressions for the velocity components for a uniform flow that is directed at an angle $\pi/2 + \Delta$.

Now

$$v = V\sin\alpha$$
$$u = V\cos\alpha$$

at $\alpha = \pi/2$, it is obvious that $v = V$ and $u = 0$. Substituting these into the relations of the previous problem produces

$$v_r = u\cos\theta + v\sin\theta = 0\cos\theta + V\sin\theta = V\sin\theta$$
$$v_\theta = -u\sin\theta + v\cos\theta = 0\ \sin\theta + V\cos\theta = V\cos\theta$$

For a uniform flow that is directed at an angle of $\alpha = \pi/2 + \Delta$

$$v = V\sin\left(\frac{\pi}{2} + \Delta\right) = V\cos\Delta$$
$$u = V\cos\left(\frac{\pi}{2} + \Delta\right) = -V\sin\Delta$$

Substituting these into the relations of the previous problem produces

$$v_r = u\cos\theta + v\sin\theta = -V\sin\Delta\cos\theta + V\cos\Delta\sin\theta = V\sin(\Delta - \theta)$$
$$v_\theta = -u\sin\theta + v\cos\theta = V\sin\Delta\sin\theta + V\cos\Delta\cos\theta = V\cos(\Delta - \theta)$$

Problem 4. – Prove that as the limit line for radial flow is approached the acceleration of the flow approaches ∞.

Flow acceleration is composed of an unsteady and convection terms

$$\frac{D\vec{V}}{Dt} = \frac{\partial V}{\partial t} + \vec{V} \bullet \nabla\vec{V}$$

Since the flow is steady and purely radial $\partial V/\partial t = 0$ and $\vec{V} = v_r\hat{e}_r$. In cylindrical coordinates

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{\partial}{r\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Now

$$\vec{V} \bullet \nabla\vec{V} = v_r\frac{\partial\vec{V}}{\partial r} = v_r\frac{\partial v_r}{\partial r}\hat{e}_r = v_r\frac{dv_r}{dr}\hat{e}_r$$

The radial derivative of the Mach number is

$$\frac{dM}{dr} = \frac{d(V/a)}{dr} = \frac{d(v_r/a)}{dr} = \frac{1}{a}\frac{dv_r}{dr} - \frac{v_r}{a^2}\frac{da}{dr} = \frac{v_r}{M}\frac{dv_r}{dr} - \frac{M}{a}\frac{da}{dr}$$

But

$$\frac{a}{a_o} = \left(\frac{T}{T_o}\right)^{\frac{1}{2}} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{2}}$$

so

$$a = a_o\left(1+\frac{\gamma-1}{2}M^2\right)^{-\frac{1}{2}}$$

Take the logarithmic derivative of this to get

$$\frac{1}{a}\frac{da}{dr} = -\frac{1}{2}\frac{(\gamma-1)M}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{dr}$$

Insert this into the expression for the radial derivative of the Mach number and obtain

$$\frac{dM}{dr} = \frac{v_r}{M}\frac{dv_r}{dr} + \frac{1}{2}\frac{(\gamma-1)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{dr}$$

Hence,

$$\frac{dM}{dr} = \frac{v_r}{M}\frac{dv_r}{dr} + \frac{1}{2}\frac{(\gamma-1)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{dr}$$

Rearranging gives

$$v_r\frac{dv_r}{dr} = \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{dr}$$

The radial derivative of the Mach number can also be obtained from Eq.(12. 7), i.e.,

$$\frac{dr}{dM} = \frac{r}{n}\frac{M^2-1}{\left(1+\frac{\gamma-1}{2}M^2\right)M}$$

Therefore,

$$v_r\frac{dv_r}{dr} = \frac{n}{r}\frac{M^2}{(M^2-1)}$$

As $r \to r^*$ $M \to 1$, hence, the convective acceleration is infinite.

Problem 5. – Starting from the concept that the streamlines are straight for radial flow and therefore have the form $\psi = c\theta = c\tan^{-1}(y/x)$, develop an equation for the velocity field of a compressible fluid.

As defined in Problem 11.3

$$u = \frac{\rho_{ref}}{\rho}\frac{\partial \psi}{\partial y}$$

$$v = -\frac{\rho_{ref}}{\rho}\frac{\partial \psi}{\partial x}$$

Now $\psi = c\tan^{-1}(y/x)$, so

$$\frac{\partial \psi}{\partial y} = c\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\frac{1}{x} = \frac{cx}{x^2 + y^2}$$

$$\frac{\partial \psi}{\partial x} = c\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\frac{-y}{x^2} = \frac{-cy}{x^2 + y^2}$$

So the velocity components are

$$u = \frac{\rho_{ref}}{\rho}\left(\frac{cx}{x^2 + y^2}\right)$$

$$v = \frac{\rho_{ref}}{\rho}\left(\frac{cy}{x^2 + y^2}\right)$$

Hence,

$$V = \sqrt{u^2 + v^2} = \frac{\rho_{ref}}{\rho}\sqrt{\frac{c^2}{x^2 + y^2}} = \frac{\rho_{ref}}{\rho}\frac{c}{r}$$

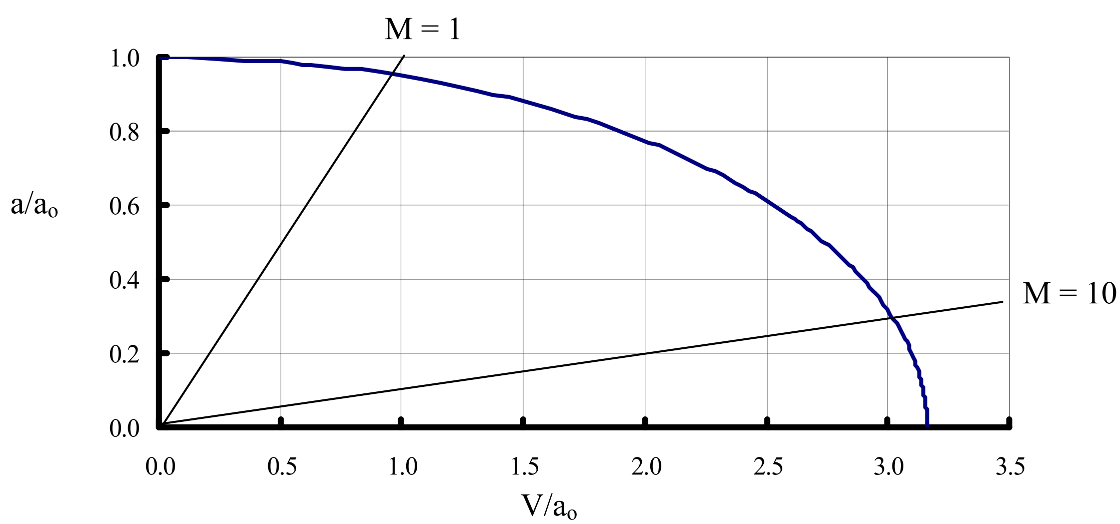
Problem 6. – Sketch the ellipse of Eq.(12.13) in the first quadrant. Indicate roughly subsonic and hypersonic regimes. Use the sketch to explain which effect, thermodynamic or inertial, dominates in subsonic and hypersonic accelerations.

Equation (12.13)

$$\frac{a^2}{(a_o^2)} + \frac{V^2}{\left(\dfrac{2}{\gamma - 1} a_o^2\right)} = 1$$

is plotted for a γ = 1.4 in the figure below. The y axis may be regarded as the thermodynamic axis since it contains the speed of sound ratio a/a_o. The x axis may be regarded as the kinematic axis since it contains the speed ratio V/a_o. Lines for two Mach numbers are shown on the plot. To the left of the M =1 line the flow is subsonic; to the right supersonic.

Because the ellipse is relatively flat in the subsonic regime, this indicates that Mach number changes within this regime are largely due to changes in V. On the other hand, because the ellipse becomes flat in the very high Mach number regime (hypersonic), this indicates that Mach number changes within this regime are largely due to changes in the thermodynamics, i.e., temperature.



Problem 7. – Develop expressions for the flow variables, T_o/T, p_o/p and ρ_o/ρ in terms of the radius, r, for the irrotational vortex.

We may start from the following

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}$$

For the irrotational compressible vortex we have shown [Eq.(12.16)] that

$$\frac{r}{r_{min}} = \sqrt{1 + \frac{2}{\gamma - 1}\frac{1}{M^2}}$$

so at M = 1

$$\frac{r^*}{r_{min}} = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$

Combining these and solving gives

$$\frac{\gamma - 1}{2} M^2 = \frac{1}{\frac{\gamma + 1}{\gamma - 1}\left(\frac{r}{r^*}\right)^2 - 1}$$

Replacing this term in each of the ratios and defining R = r/r* produces the desired expressions

$$\frac{T_o}{T} = \frac{(\gamma - 1)}{(\gamma - 1) - (\gamma + 1)R^2}$$

$$\frac{p_o}{p} = \left(\frac{(\gamma - 1)}{(\gamma - 1) - (\gamma + 1)R^2}\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{(\gamma - 1)}{(\gamma - 1) - (\gamma + 1)R^2}\right)^{\frac{1}{\gamma - 1}}$$

Problem 8. – Plot the incompressible flow equiangular spiral from $\theta = 0$ to $\theta = 4\pi$. Assume that the constant c is unity, i.e., C = 0. Select the ratio of Q/Γ so that r = 4 at $\theta = 4\pi$.

The equation for an incompressible flow equiangular spiral is

$$r = e^{\frac{Q\theta - C}{\Gamma}} = ce^{\frac{Q}{\Gamma}\theta} = e^{\frac{Q}{\Gamma}\theta}$$

At $\theta = 4\pi$ r = 4 therefore,

$$4 = e^{\frac{Q}{\Gamma}4\pi}$$

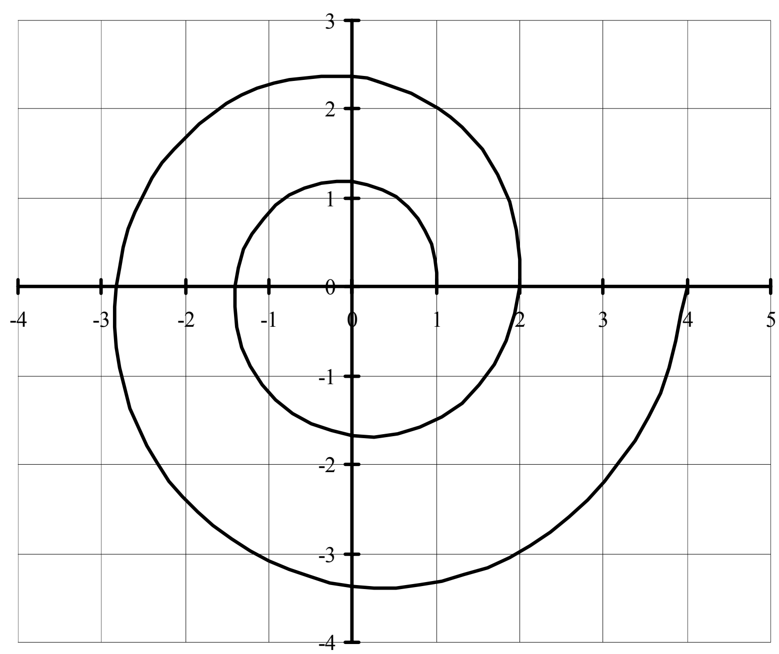
So

$$\frac{Q}{\Gamma} = \frac{1}{4\pi}\ln 4$$

Hence

$$r = 4^{\frac{\theta}{4\pi}}$$

The corresponding x and y coordinates are $x = r\cos\theta$ and $y = r\sin\theta$. A plot of these with θ as the parameter provides the following



Problem 9. – Use the hodograph transformation to obtain a solution for the case in which $\psi = \psi(V)$. Show that for this case $\phi = c\alpha$.

Elimination of the velocity potential from the Chaplygin-Molenbroek equations, Eqs.(12.52) and (12.53) was shown to produce

$$V\frac{\partial}{\partial V}\left(\frac{\rho_o}{\rho}V\frac{\partial\psi}{\partial V}\right)+\frac{\rho_o}{\rho}\left(1-\frac{V^2}{a^2}\right)\frac{\partial^2\psi}{\partial\alpha^2}=0$$

Since for this problem $\psi = \psi(V)$, this expression reduces to

$$\frac{d}{dV}\left(\frac{\rho_o}{\rho}V\frac{d\psi}{dV}\right)=0$$

Integration brings

$$\frac{\rho_o}{\rho}V\frac{d\psi}{dV}=C$$

But from Eq.(12.52)

$$\frac{\partial\phi}{\partial\alpha}=\frac{\rho_o}{\rho}V\frac{d\psi}{dV}=C$$

Hence,

$$\phi = C\alpha$$

Problem 10. – Another method to obtain the hodograph equations is to make use of the *Legrendre transformation*, Zwillinger, Ref. 24. In this approach a function, $\Phi(u,v)$, in the hodograph plane is related to the velocity potential, $\phi(x,y)$, by

$$\Phi = xu + yv - \phi$$

Use this relation to show that

(a) $\dfrac{\partial\Phi}{\partial u}=x$ and $\dfrac{\partial\Phi}{\partial v}=y$

(b) $$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} = \dfrac{1}{J}\dfrac{\partial^2\Phi}{\partial v^2} \\[2ex] \dfrac{\partial^2\phi}{\partial x\partial y} = -\dfrac{1}{J}\dfrac{\partial^2\Phi}{\partial u\partial v} \\[2ex] \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{J}\dfrac{\partial^2\Phi}{\partial u^2} \end{cases}$$

where $J = \begin{vmatrix} \Phi_{uu} & \Phi_{uv} \\ \Phi_{uv} & \Phi_{vv} \end{vmatrix}$.

(c) Finally, show that Eq.(12.27) is transformed into the following linear equation

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\Phi}{\partial v^2} + 2\frac{uv}{a^2}\frac{\partial^2\Phi}{\partial u\partial v} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\Phi}{\partial u^2} = 0$$

(a) Differentiate $\Phi = xu + yv - \phi(x, y)$ with respect to u and obtain

$$\frac{\partial\Phi}{\partial u} = u\frac{\partial x}{\partial u} + x + v\frac{\partial y}{\partial u} - \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} - \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}$$

But from the definition of the velocity potential $\dfrac{\partial\phi}{\partial x} = u$ and $\dfrac{\partial\phi}{\partial y} = v$, so the above reduces to

$$\frac{\partial\Phi}{\partial u} = u\frac{\partial x}{\partial u} + x + v\frac{\partial y}{\partial u} + -\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} - \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u} = u\frac{\partial x}{\partial u} + x + v\frac{\partial y}{\partial u} + -u\frac{\partial x}{\partial u} - v\frac{\partial y}{\partial u} = x$$

Similarly differentiate $\Phi = xu + yv - \phi(x, y)$ with respect to v and obtain

$$\frac{\partial\Phi}{\partial v} = u\frac{\partial x}{\partial v} + y + v\frac{\partial y}{\partial v} - \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} - \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v} = u\frac{\partial x}{\partial v} + y + v\frac{\partial y}{\partial v} - u\frac{\partial x}{\partial v} - v\frac{\partial y}{\partial v} = y$$

(b) Differentiate $\partial\Phi/\partial u = x$ first with respect to u and also with respect to v to obtain

$$\Phi_{uu} = x_u$$
$$\Phi_{uv} = x_v$$

Similarly using $\partial\Phi/\partial v = y$ we obtain

$$\Phi_{vu} = y_u$$
$$\Phi_{vv} = y_v$$

Now use the chain rule and write

$$1 = \frac{\partial u}{\partial u} = \frac{\partial(\phi_x)}{\partial u} = \frac{\partial(\phi_x)}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial(\phi_x)}{\partial y}\frac{\partial y}{\partial u} = \phi_{xx}\Phi_{uu} + \phi_{xy}\Phi_{vu} = 1$$

Likewise

$$0 = \frac{\partial u}{\partial v} = \frac{\partial(\phi_x)}{\partial v} = \frac{\partial(\phi_x)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial(\phi_x)}{\partial y}\frac{\partial y}{\partial v} = \phi_{xx}\Phi_{uv} + \phi_{xy}\Phi_{vv} = 0$$

The above pair may be written in matrix-vector form as

$$\begin{bmatrix} \Phi_{uu} & \Phi_{vu} \\ \Phi_{uv} & \Phi_{vv} \end{bmatrix} \begin{bmatrix} \phi_{xx} \\ \phi_{xy} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving this for ϕ_{xx} and ϕ_{xy} yields

$$\phi_{xx} = \frac{\begin{vmatrix} 1 & \Phi_{vu} \\ 0 & \Phi_{vv} \end{vmatrix}}{\begin{vmatrix} \Phi_{uu} & \Phi_{vu} \\ \Phi_{uv} & \Phi_{vv} \end{vmatrix}} = \frac{\Phi_{vv}}{\begin{vmatrix} \Phi_{uu} & \Phi_{vu} \\ \Phi_{uv} & \Phi_{vv} \end{vmatrix}}$$

$$\phi_{xy} = \frac{\begin{vmatrix} \Phi_{uu} & 1 \\ \Phi_{uv} & 0 \end{vmatrix}}{\begin{vmatrix} \Phi_{uu} & \Phi_{vu} \\ \Phi_{uv} & \Phi_{vv} \end{vmatrix}} = \frac{-\Phi_{uv}}{\begin{vmatrix} \Phi_{uu} & \Phi_{vu} \\ \Phi_{uv} & \Phi_{vv} \end{vmatrix}}$$

Following the same procedure

$$1 = \frac{\partial v}{\partial v} = \frac{\partial(\phi_y)}{\partial v} = \frac{\partial(\phi_y)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial(\phi_y)}{\partial y}\frac{\partial y}{\partial v} = \phi_{yx}\Phi_{uv} + \phi_{yy}\Phi_{vv} = 1$$

Likewise

$$0=\frac{\partial v}{\partial u}=\frac{\partial(\phi_x)}{\partial u}=\frac{\partial(\phi_y)}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial(\phi_y)}{\partial y}\frac{\partial y}{\partial u}=\phi_{yx}\Phi_{uu}+\phi_{yy}\Phi_{vu}=0$$

The above pair may be written in matrix-vector form as

$$\begin{bmatrix}\Phi_{uu} & \Phi_{vu}\\ \Phi_{uv} & \Phi_{vv}\end{bmatrix}\begin{bmatrix}\phi_{yx}\\ \phi_{yy}\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$$

Solving this for ϕ_{yx} and ϕ_{yy} yields

$$\phi_{yy}==\frac{\Phi_{uu}}{\begin{vmatrix}\Phi_{uu} & \Phi_{vu}\\ \Phi_{uv} & \Phi_{vv}\end{vmatrix}}$$

$$\phi_{yx}=\frac{-\Phi_{vu}}{\begin{vmatrix}\Phi_{uu} & \Phi_{vu}\\ \Phi_{uv} & \Phi_{vv}\end{vmatrix}}$$

Because of irrotationality

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0=\phi_{yx}-\phi_{xy}$$

So $\phi_{xy}=\phi_{yx}$. Accordingly, from the above we see that

$$\phi_{xy}=\frac{-\Phi_{uv}}{\begin{vmatrix}\Phi_{uu} & \Phi_{vu}\\ \Phi_{uv} & \Phi_{vv}\end{vmatrix}}=\phi_{yx}=\frac{-\Phi_{vu}}{\begin{vmatrix}\Phi_{uu} & \Phi_{vu}\\ \Phi_{uv} & \Phi_{vv}\end{vmatrix}}$$

or $\Phi_{uv}=\Phi_{vu}$. Thus the denominators may be written as

$$J=\begin{vmatrix}\Phi_{uu} & \Phi_{uv}\\ \Phi_{uv} & \Phi_{vv}\end{vmatrix}$$

Finally then we obtain the expressions requested i.e.,

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{J}\frac{\partial^2 \Phi}{\partial v^2}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{1}{J}\frac{\partial^2 \Phi}{\partial u \partial v}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{J}\frac{\partial^2 \Phi}{\partial u^2}$$

(c) Equation (12.27)

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x} - \frac{uv}{a^2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y} = 0$$

may be written as

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2 \phi}{\partial x^2} - \frac{uv}{a^2}\left(\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right) + \left(1-\frac{v^2}{a^2}\right)\frac{\partial^2 \phi}{\partial y^2} =$$

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2 \phi}{\partial x^2} - 2\frac{uv}{a^2}\frac{\partial^2 \phi}{\partial x \partial y} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial^2 \phi}{\partial y^2} = 0$$

The derivatives of the potential may be replaced by using the expressions from part (b)

$$\left(1-\frac{u^2}{a^2}\right)\frac{1}{J}\frac{\partial^2 \Phi}{\partial v^2} - 2\frac{uv}{a^2}\frac{1}{J}\frac{\partial^2 \Phi}{\partial u \partial v} + \left(1-\frac{v^2}{a^2}\right)\frac{1}{J}\frac{\partial^2 \Phi}{\partial u^2} = 0$$

Assuming that $J \neq 0$ this becomes

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2 \Phi}{\partial v^2} + 2\frac{uv}{a^2}\frac{\partial^2 \Phi}{\partial u \partial v} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial^2 \Phi}{\partial u^2} = 0$$

Problem 11. – Prove Eq.(12.49).

Start with Eq.(12.48)

$$\frac{\rho_o}{\rho} = \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-\frac{1}{\gamma-1}}$$

Differentiate this with respect to V, i.e.,

$$\frac{d}{dV}\left(\frac{\rho_o}{\rho}\right) = -\frac{1}{\gamma - 1}\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-\frac{1}{\gamma-1}-1}\left[-(\gamma - 1)\frac{V}{a_o^2}\right]$$

$$= \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-\frac{1}{\gamma-1}}\left(\frac{V}{a_o^2}\right)\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-1}$$

$$= \left(\frac{\rho_o}{\rho}\right)\left(\frac{V}{a_o^2}\right)\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-1}$$

But $a_o^2 = a^2 + \frac{\gamma - 1}{2}V^2$ or

$$1 = \frac{a^2}{a_o^2} + \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}$$

Hence,

$$\frac{a_o^2}{a^2} = \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-1}$$

Finally then

$$\frac{d}{dV}\left(\frac{\rho_o}{\rho}\right) = \left(\frac{\rho_o}{\rho}\right)\left(\frac{V}{a_o^2}\right)\left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_o^2}\right)^{-1} = \left(\frac{\rho_o}{\rho}\right)\left(\frac{V}{a_o^2}\right)\left(\frac{a_o^2}{a^2}\right)$$

$$= \left(\frac{\rho_o}{\rho}\right)\left(\frac{V}{a^2}\right)$$

Problem 12. – In the solution of problems using the hodograph equations, the resulting expressions often contain the following two parameters: $\tau = (V/V_{max})^2$ and $\beta = 1/(\gamma - 1)$. Show that

(a) $\dfrac{\rho}{\rho_o} = (1-\tau)^{\beta}$

(b) $M^2 = \dfrac{2\beta\tau}{1-\tau}$.

From $a_o^2 = a^2 + \dfrac{\gamma-1}{2}V^2$ when a = 0, V = V_{max}; therefore $V_{max} = \left(\sqrt{\dfrac{2}{\gamma - 1}}\right)a_o$.

(a) So

$$\left(\frac{V}{V_{max}}\right)^2 = \tau = 1 - \left(\frac{a}{a_o}\right)^2$$

But

$$\left(\frac{a}{a_o}\right)^2 = \frac{T}{T_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma-1}$$

Hence,

$$\tau = 1 - \left(\frac{\rho}{\rho_o}\right)^{\gamma-1}$$

Finally then with $\beta = 1/(\gamma - 1)$

$$\frac{\rho}{\rho_o} = (1-\tau)^{\beta}$$

(b) From static to total density relation and the result of part (a), we have

$$\left(\frac{\rho_o}{\rho}\right) = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{1}{2\beta}M^2\right)^{\beta} = \frac{1}{(1-\tau)^{\beta}}$$

Thus,

$$1 + \frac{1}{2\beta}M^2 = \frac{1}{1-\tau}$$

and so

$$M^2 = 2\beta\left(\frac{1}{1-\tau} - 1\right) = \frac{2\beta\tau}{1-\tau}$$

Problem 13. – Using Eq. (12.79), show that Eq.(12.77) can be written as

$$\left(\frac{dv_r}{d\theta}\right)^2\left(v_r+\frac{d^2v_r}{d\theta^2}\right)=\frac{\gamma-1}{2}\left[V_{max}^2-v_r^2-\left(\frac{dv_r}{d\theta}\right)^2\right]\left(\frac{d^2v_r}{d\theta^2}+\cot\theta\frac{dv_r}{d\theta}+2v_r\right)$$

Verify that this is identically satisfied by a uniform stream given by $v_r = V\cos\theta$.

Equation (12.77), the *Taylor-Maccoll equation* is

$$\left[a^2-\left(\frac{dv_r}{d\theta}\right)^2\right]\frac{d^2v_r}{d\theta^2}+\left(a^2\cot\theta\right)\frac{dv_r}{d\theta}+\left[2a^2-\left(\frac{dv_r}{d\theta}\right)^2\right]v_r=0$$

Rearranging

$$\left(\frac{dv_r}{d\theta}\right)^2\left(\frac{d^2v_r}{d\theta^2}+v_r\right)-a^2\frac{d^2v_r}{d\theta^2}-a^2\cot\theta\frac{dv_r}{d\theta}-2a^2v_r=0$$

or

$$\left(\frac{dv_r}{d\theta}\right)^2\left(\frac{d^2v_r}{d\theta^2}+v_r\right)=a^2\left(\frac{d^2v_r}{d\theta^2}+\cot\theta\frac{dv_r}{d\theta}+2v_r\right)$$

Next combine Eq.(12.78)

$$v_\theta=\frac{dv_r}{d\theta}$$

and Eq.(12.79)

$$a^2=\frac{\gamma-1}{2}(V_{max}^2-V^2)=\frac{\gamma-1}{2}\left[V_{max}^2-\left(v_r^2+v_\theta^2\right)\right]$$

$$=\frac{\gamma-1}{2}\left\{V_{max}^2-\left[v_r^2+\left(\frac{dv_r}{d\theta}\right)^2\right]\right\}$$

Using this to replace the a^2 in the above gives the desired expression

$$\left(\frac{dv_r}{d\theta}\right)^2\left(v_r+\frac{d^2v_r}{d\theta^2}\right)=\frac{\gamma-1}{2}\left[V_{max}^2-v_r^2-\left(\frac{dv_r}{d\theta}\right)^2\right]\left(\frac{d^2v_r}{d\theta^2}+\cot\theta\frac{dv_r}{d\theta}+2v_r\right)$$

Now suppose $v_r = V\cos\theta$. So $dv_r/d\theta = -V\sin\theta$ and $d^2v_r/d\theta^2 = -V\cos\theta$. Substitution into the above brings

$$(-V\sin\theta)^2(V\cos\theta - V\cos\theta)$$

$$= \frac{\gamma-1}{2}\left[V_{max}^2 - V^2\cos^2\theta - (-V\sin\theta)^2\right]\left[-V\cos\theta + \cot\theta(-V\sin\theta) + 2V\cos\theta\right]$$

or

$$V^2\sin^2\theta(0) = \frac{\gamma-1}{2}\left(V_{max}^2 - V^2\right)(0)$$

$$0 = 0$$

Problem 14. – A steady, two-dimensional, supersonic flow, uniformly streams along a horizontal wall that is aligned with the x-axis. The stream encounters a sharp corner located at x = 0. Show that the maximum angle through which the flow may turn is given by $\pi/2b - \pi/2$, where $b^2 = (\gamma - 1)/(\gamma + 1)$. Also show that the maximum angle can only occur if the original flow is sonic.

From Eq.(12.66)

$$\nu = (\theta + \theta_{ref}) - \left(\frac{\pi}{2} - \mu\right) = \frac{1}{b}\tan^{-1}\left(b\sqrt{M^2-1}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right)$$

$$= \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left(\sqrt{\frac{\gamma-1}{\gamma+1}}\sqrt{M^2-1}\right) - \tan^{-1}\left(\sqrt{M^2-1}\right)$$

The maximum value of ν will occur at $M \to \infty$. As Mach number goes to infinity the argument of the inverse tangent becomes infinite, i.e., $\tan^{-1}(\infty) = \pi/2$. Therefore, the above becomes

$$\nu_{max} = \frac{1}{b}\frac{\pi}{2} - \frac{\pi}{2}$$

The angle through a flow is turned may be expressed as

$$\Delta = \nu_2 - \nu_1$$

The maximum turning angle will occur when $\nu_2 = \nu_{max}$ and $\nu_1 = 0$. As shown above ν_{max} occurs when $M_2 \to \infty$; The upstream Prandtl-Meyer function, ν_1, i.e.,

$$\nu_1 = \frac{1}{b}\tan^{-1}\left(b\sqrt{M_1^2 - 1}\right) - \tan^{-1}\left(\sqrt{M_1^2 - 1}\right)$$

is seen to vanish when $M_1 = 1$.

Problem 15. – Uniform supersonic flow at Mach 3.0 and p = 20 kPa passes over a cone of semi-vertex angle of 20° aligned parallel to the flow direction. Determine the shock wave angle, the Mach number of the flow along the cone surface, and the surface pressure. Take $\gamma = 1.3$.

Except for the ratio of specific heats this is identical to Example 12.8.

Iteration of shock angles is used until a value of $\theta_s = 29.24443^\circ$ produces the desired cone half angle of $\theta_c = 20^\circ$. At the free stream Mach number, $M_1 = 3$, and this shock angle, using Eqs.(12.85), (12.86), (12.87), (12.81) and (12.82), we find M_2, V_2, δ_s, v_r and v_θ, respectively

M_2	$\tan(\delta_s)$	δ_s (deg)	V_2	v_r	v_θ
2.4641	0.2228	12.5614	0.6904	0.6613	0.1982

Spreadsheet calculation results for the first five increments of $\Delta\theta = 0.1^\circ$ for $\theta_s = 29.24443^\circ$, $M_1 = 3$ and $\gamma = 1.3$ are as follows

No.	θ (deg)	$(v_r)_p$	$F[(v_r)_i,(v_\theta)_i]$	$(v_\theta)_p$	$F[(v_r)_p,(v_\theta)_p]$	$(v_r)_{i+1}$	$(v_\theta)_{i+1}$	V	M	δ (rad)
1	29.2444	0.6613		-0.1982		0.6613	-0.1982	0.6904	2.4641	-0.2912
2	29.1444	0.6617	-1.2765	-0.1960	-1.2686	0.6617	-0.1960	0.6901	2.4620	-0.2879
3	29.0444	0.6620	-1.2686	-0.1938	-1.2612	0.6620	-0.1938	0.6898	2.4600	-0.2847
4	28.9444	0.6624	-1.2612	-0.1916	-1.2545	0.6624	-0.1916	0.6895	2.4580	-0.2815
5	28.8444	0.6627	-1.2545	-0.1894	-1.2482	0.6627	-0.1894	0.6892	2.4561	-0.2784
6	28.7444	0.6630	-1.2482	-0.1872	-1.2424	0.6630	-0.1872	0.6889	2.4542	-0.2752

Spreadsheet calculation results near the cone surface for $\Delta\theta = 0.1^\circ$, $\theta_s = 29.24443^\circ$, $M_1 = 3$ and $\gamma = 1.3$ are contained in the following table

No.	θ (deg)	$(v_r)_p$	$F[(v_r)_i,(v_\theta)_i]$	$(v_\theta)_p$	$F[(v_r)_p,(v_\theta)_p]$	$(v_r)_{i+1}$	$(v_\theta)_{i+1}$	V	M	δ (rad)
92	20.1444	0.6775	-1.3397	-0.0034	-1.3459	0.6775	-0.0034	0.6775	2.3786	-0.0050
93	20.0444	0.6775	-1.3459	-0.0011	-1.3522	0.6775	-0.0011	0.6775	2.3786	-0.0016
94	19.9444	0.6775	-1.3522	0.0013	-1.3587	0.6775	0.0013	0.6775	2.3786	0.0019
95	19.8444	0.6775	-1.3587	0.0037	-1.3654	0.6775	0.0037	0.6775	2.3786	0.0054

Since the velocity at the surface is equal to the radial velocity component, we may readily compute the Mach number from Eq.(12.84) and the static pressure on the surface.

θ_c (deg)	M_c	V_c	v_r	v_θ	p_s
20.0000	2.3786	0.6775	0.6775	0.0000	53.0308

Problem 16. – Uniform supersonic flow at Mach 4.0 and p = 20 kPa passes over a cone of semi-vertex angle of 20° aligned parallel to the flow direction. Determine the shock wave angle, the Mach number of the flow along the cone surface, and the surface pressure. Take $\gamma = 1.4$.

Except for the upstream Mach number this is identical to Example 12.8.

Iteration of shock angles continued until a value of $\theta_s = 26.4850^\circ$ produced the desired cone half angle of $\theta_c = 20^\circ$. At the free stream Mach number, $M_1 = 4$, and this shock angle, using Eqs.(12.85), (12.86), (12.87), (12.81) and (12.82), we find M_2, V_2, δ_s, v_r and v_θ, respectively

M_2	$\tan(\delta_s)$	δ_s (deg)	V_2	v_r	v_θ
2.9698	0.2574	0.2519	0.7989	0.7813	0.1668

Calculation results near the cone surface for $\Delta\theta = 0.1^\circ$, $\theta_s = 26.4850^\circ$, $M_1 = 4$ and $\gamma = 1.4$.

No.	θ (deg)	$(v_r)_p$	$F[(v_r)_i,(v_\theta)_i]$	$(v_\theta)_p$	$F[(v_r)_p,(v_\theta)_p]$	$(v_r)_{i+1}$	$(v_\theta)_{i+1}$	V	M	δ (rad)
64	20.1850	0.7908	-1.5610	-0.0051	-1.5680	0.7908	-0.0051	0.7908	2.8891	-0.0064
65	20.0850	0.7908	-1.5680	-0.0023	-1.5752	0.7908	-0.0023	0.7908	2.8890	-0.0030
66	19.9850	0.7908	-1.5752	0.0004	-1.5827	0.7908	0.0004	0.7908	2.8890	0.0005
67	19.8850	0.7908	-1.5827	0.0032	-1.5905	0.7908	0.0032	0.7908	2.8890	0.0040

Since the velocity at the surface is equal to the radial velocity component, we may readily compute the Mach number on the surface from Eq.(12.84).

θ_c (deg)	M_c	V_c	v_r	v_θ	p_s
20.0000	2.8890	0.7908	0.7908	0.0000	80.12361

Problem 17. – Uniform supersonic flow at Mach 3.0 and p = 20 kPa passes over a cone of semi-vertex angle of 30° aligned parallel to the flow direction. Determine the shock wave angle, the Mach number of the flow along the cone surface, and the surface pressure. Take $\gamma = 1.4$.

Except for the cone angle this is identical to Example 12.8.

Iteration of shock angles continued until a value of $\theta_s = 39.7841^\circ$ produced the desired cone half angle of $\theta_c = 30^\circ$. At the free stream Mach number, $M_1 = 3$, and this shock angle, using Eqs.(12.85), (12.86), (12.87), (12.81) and (12.82), we find M_2, V_2, δ_s, v_r and v_θ, respectively

M_2	$\tan(\delta_s)$	δ_s (deg)	V_2	v_r	v_θ
1.9038	0.3974	0.3783	0.6483	0.6161	0.2015

Calculation results near the cone surface for $\Delta\theta = 0.1^\circ$ for $\theta_s = 39.7841^\circ$, $M_1 = 3$ and $\gamma = 1.4$ are as follows

No.	θ (deg)	$(v_r)_p$	$F[(v_r)_i,(v_\theta)_i]$	$(v_\theta)_p$	$F[(v_r)_p,(v_\theta)_p]$	$(v_r)_{i+1}$	$(v_\theta)_{i+1}$	V	M	δ (rad)
97	30.1841	0.6335	-1.2566	-0.0041	-1.2602	0.6335	-0.0041	0.6336	1.8310	-0.0064
98	30.0841	0.6335	-1.2602	-0.0019	-1.2639	0.6335	-0.0019	0.6335	1.8310	-0.0029
99	29.9841	0.6335	-1.2639	0.0003	-1.2677	0.6335	0.0004	0.6335	1.8310	0.0006
100	29.8841	0.6335	-1.2677	0.0026	-1.2716	0.6335	0.0026	0.6335	1.8310	0.0041

Since the velocity at the surface is equal to the radial velocity component, we may readily compute the Mach number on the surface from Eq.(12.84).

θ_c (deg)	M_c	V_c	v_r	v_θ	p_s
30.0000	1.8310	0.6335	0.6335	0.0000	92.4554

Problem 18. – A supersonic diffuser contains a conical spike of semi-vertex angle 5°; the spike is aligned with the flow (Figure P12.18). Determine the Mach number of the flow along the cone surface and the static pressure at the surface of the cone. Altitude = 5 km and $\gamma = 1.4$.

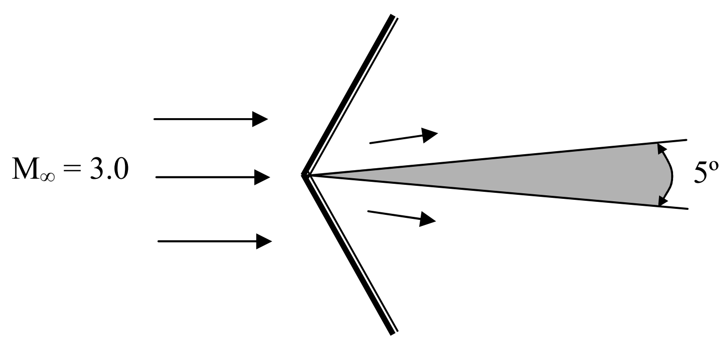


Figure P12.18

At an altitude of 5 km the local static pressure is 54.05 kPa.

Iteration of shock angles continued until a value of $\theta_s = 19.75086^\circ$ produced the desired cone half angle of $\theta_c = 5^\circ$. At the free stream Mach number, $M_1 = 3$, and this shock angle, using Eqs.(12.85), (12.86), (12.87), (12.81) and (12.82), we find M_2, V_2, δ_s, v_r and v_θ, respectively

M_2	$\tan(\delta_s)$	δ_s (deg)	V_2	v_r	v_θ
2.9788	0.0072	0.0072	0.7997	0.7546	0.2648

Calculation results near the cone surface for $\Delta\theta = 0.1^\circ$ for $\theta_s = 19.75086^\circ$, $M_1 = 3$ and $\gamma = 1.4$ are as follows

No.	θ (deg)	$(v_r)_p$	$F[(v_r)_i,(v_\theta)_i]$	$(v_\theta)_p$	$F[(v_r)_p,(v_\theta)_p]$	$(v_r)_{i+1}$	$(v_\theta)_{i+1}$	V	M	δ (rad)
145	5.3509	0.7904	-1.5086	-0.0043	-1.5351	0.7904	-0.0043	0.7904	2.8851	-0.0054
146	5.2509	0.7904	-1.5353	-0.0016	-1.5634	0.7904	-0.0016	0.7904	2.8851	-0.0020
147	5.1509	0.7904	-1.5637	0.0012	-1.5936	0.7904	0.0012	0.7904	2.8851	0.0015
148	5.0509	0.7904	-1.5939	0.0040	-1.6258	0.7904	0.0040	0.7904	2.8851	0.0051

Since the velocity at the surface is equal to the radial velocity component, we may readily compute the Mach number on the surface from Eq.(12.84).

θ_c (deg)	M_c	V_c	v_r	v_θ	p_s
5.1938	2.8851	0.7904	0.7904	0.0000	64.2781

Chapter Thirteen

LINEARIZED FLOWS

Problem 1. – The lift coefficient versus angle of attack for an airfoil, as measured in a low-speed wind tunnel, is given in Figure P13.1. Sketch this curve for the same airfoil at a Mach number of 0.45.

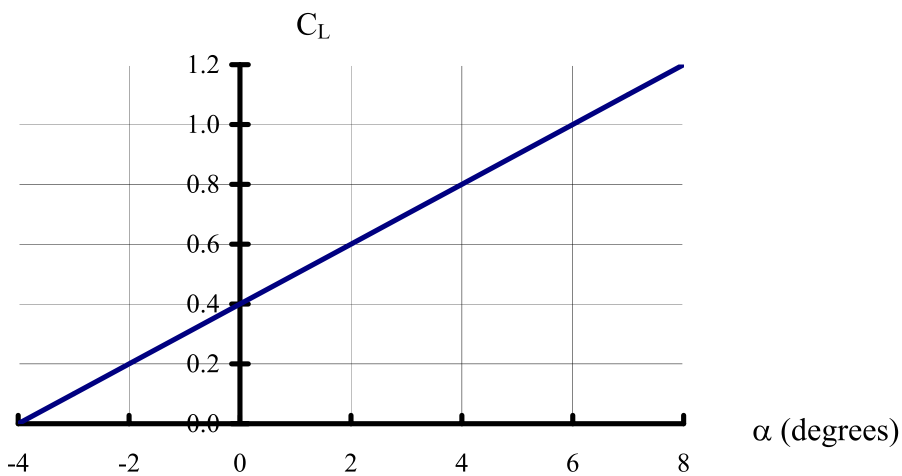


Figure P13.1

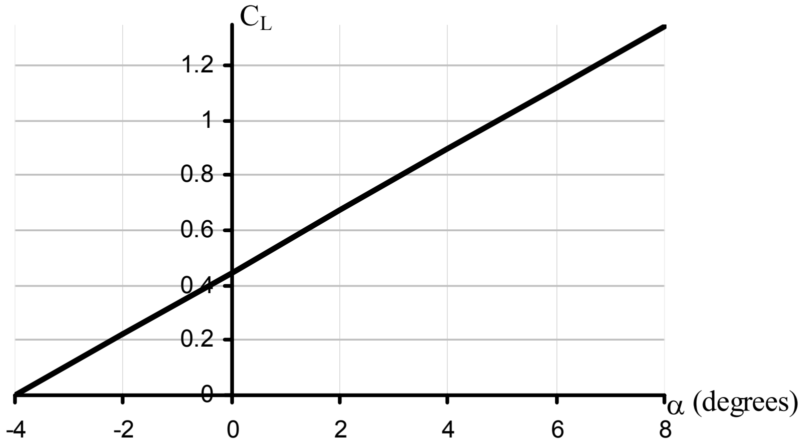
From the Prandtl Glauert similarity rule, we can write

$$C_{L\,M_\infty=0} = C_L\sqrt{1-M_\infty^2}$$

Therefore,

$$C_L = \frac{C_{L\,M_\infty=0}}{\sqrt{1-M_\infty^2}} = \frac{C_{L\,M_\infty=0}}{\sqrt{1-0.45^2}} = 1.1198\, C_{L\,M_\infty=0}$$

Using this result and the values of $C_{L\,M_\infty=0}$ from Figure P13.1 we can sketch the lift coefficient versus angle of attack for the same airfoil at a Mach number of 0.45



Problem 2. – Using the potential equation

$$\frac{\partial^2 \phi_p}{\partial x^2}\left(1 - M_\infty^2\right) + \frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial z^2} = 0$$

develop the Goethert similarity rules for three-dimensional potential subsonic flow.

The equation

$$\frac{\partial^2 \phi_p}{\partial x^2}\left(1 - M_\infty^2\right) + \frac{\partial^2 \phi_p}{\partial y^2} + \frac{\partial^2 \phi_p}{\partial z^2} = 0$$

is for small-perturbation, linearized compressible three-dimensional flow. We transform this flow to an incompressible flow. Let

$$\begin{aligned} x_i &= k_1 x \\ y_i &= k_2 y \\ z_i &= k_3 z \\ \phi_i &= k_4 \phi \\ U_{\infty i} &= k_5 U_\infty \end{aligned}$$

and substitute into the potential equation:

$$\frac{k_1^2}{k_4}\left(1 - M_\infty^2\right)\frac{\partial^2 \phi_i}{\partial x_i^2} + \frac{k_2^2}{k_4}\frac{\partial^2 \phi_i}{\partial y_i^2} + \frac{k_3^2}{k_4}\frac{\partial^2 \phi_i}{\partial z_i^2} = 0$$

Multiplying this equation with k_4/k_2k_3 we obtain:

$$\frac{k_1^2}{k_2k_3}\left(1-M_\infty^2\right)\frac{\partial^2\phi_i}{\partial x_i^2}+\frac{k_2}{k_3}\frac{\partial^2\phi_i}{\partial y_i^2}+\frac{k_3}{k_2}\frac{\partial^2\phi_i}{\partial z_i^2}=0$$

In order to transform the potential equation for compressible flow into Laplace's equation, it follows that

$$\frac{k_1}{\sqrt{k_2k_3}}=\frac{1}{\sqrt{1-M_\infty^2}}$$

$$\frac{k_2}{k_3}=1$$

The boundary conditions for the three-dimensional compressible flow are:

$$\left(\frac{dy}{dx}\right)_b=\frac{v_p}{U_\infty}$$

$$\left(\frac{dz}{dx}\right)_b=\frac{w_p}{U_\infty}$$

where

$$v_p=\frac{\partial\phi_p}{\partial y}$$

$$w_p=\frac{\partial\phi_p}{\partial z}$$

Transforming the boundary conditions to the incompressible flow, we have:

$$\frac{k_1}{k_2}\left(\frac{dy_i}{dx_i}\right)_b=\frac{1}{U_{\infty i}}\frac{\partial\phi_{ip}}{\partial y_i}\frac{k_2k_5}{k_4}$$

$$\frac{k_1}{k_3}\left(\frac{dz_i}{dx_i}\right)_b=\frac{1}{U_{\infty i}}\frac{\partial\phi_{ip}}{\partial z_i}\frac{k_3k_5}{k_4}$$

To satisfy the boundary conditions for the incompressible flow, it is necessary that

$$\frac{k_1}{k_2}=\frac{k_2k_5}{k_4}$$

$$\frac{k_1}{k_3}=\frac{k_3k_5}{k_4}$$

For the incompressible flow, the Bernoulli's equation is:

$$p_{\infty i} + \frac{1}{2}\rho U_{\infty i}^2 = p_i + \frac{1}{2}\rho\left[\left(U_{\infty i} + u_{pi}\right)^2 + v_{pi}^2 + w_{pi}^2\right]$$

or

$$p_i - p_{\infty i} = \frac{1}{2}\rho U_{\infty i}^2\left[-\frac{2u_{pi}}{U_{\infty i}} - \left(\frac{u_{pi}}{U_{\infty i}}\right)^2 - \left(\frac{v_{pi}}{U_{\infty i}}\right)^2 - \left(\frac{w_{pi}}{U_{\infty i}}\right)^2\right]$$

Introducing this equation into

$$C_{p_i} = \frac{p_i - p_{\infty i}}{\frac{1}{2}\rho U_{\infty i}^2}$$

and dropping the smaller terms, we receive

$$C_{p_i} = -\frac{2u_{pi}}{U_{\infty i}}$$

For the compressible flow we have

$$C_p = -\frac{2u_p}{U_\infty} = -\frac{2\frac{\partial \phi_p}{\partial x}}{U_\infty} = -\frac{2\frac{\partial \phi_{pi}}{\partial x_i}}{U_{\infty i}}\frac{k_1 k_5}{k_4} = \frac{k_1 k_5}{k_4} C_{pi}$$

or

$$C_p = \frac{1}{1 - M_\infty^2} C_{pi}$$

Problem 3. – Tests run at $M_\infty = 0.3$ show that the lift coefficient versus angle of attack for an airfoil is given by $C_L = 0.1(\alpha + 1)$ with α in degrees. Using the appropriate similarity laws, derive an expression for C_L versus α for this airfoil at $M_\infty = 0.5$.

Using the Prandtl Glauert similarity rule

$$C_{L\, M_\infty = 0} = C_L\sqrt{1 - M_\infty^2}$$

we obtain

$$C_{L\,M_\infty=0} = C_{L\,M_\infty=0.3}\sqrt{1-0.3^2}$$

and

$$C_{L\,M_\infty=0} = C_{L\,M_\infty=0.5}\sqrt{1-0.5^2}$$

Therefore,

$$C_{L\,M_\infty=0.5} = C_{L\,M_\infty=0.3}\frac{\sqrt{1-0.3^2}}{\sqrt{1-0.5^2}} = C_{L\,M_\infty=0.3}\,1.10$$

But

$$C_{L\,M_\infty=0.3} = 0.1(\alpha+1)$$

and the expression for C_L versus α at $M_\infty = 0.5$ is

$$C_{L\,M_\infty=0.5} = 0.1(\alpha+1)1.10 = 0.11(\alpha+1)$$

Problem 4. – For the airfoil of Problem 3, plot C_L versus M_∞ from $M_\infty = 0$ to $M_\infty = 0.60$ at angles of attack of 0°, 2° and 4°.

From the Prandtl Glauert similarity rule we have

$$C_{L\,M_\infty=0} = C_L\sqrt{1-M_\infty^2} = C_{L\,M_\infty=0.3}\sqrt{1-0.3^2}$$

or

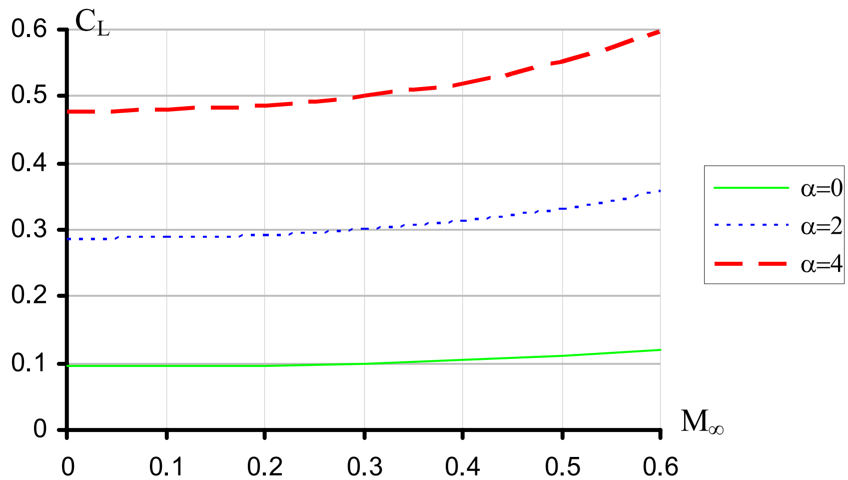
$$C_L = C_{L\,M_\infty=0.3}\frac{\sqrt{1-0.3^2}}{\sqrt{1-M_\infty^2}} = 0.1(\alpha+1)\frac{\sqrt{1-0.3^2}}{\sqrt{1-M_\infty^2}} = 0.095394\frac{(\alpha+1)}{\sqrt{1-M_\infty^2}}$$

Therefore,

$$C_{L\,\alpha=0^\circ} = 0.095394\frac{1}{\sqrt{1-M_\infty^2}}$$

$$C_{L\,\alpha=2^\circ} = 0.28618\frac{1}{\sqrt{1-M_\infty^2}}$$

$$C_{L\,\alpha=4^\circ} = 0.47697 \frac{1}{\sqrt{1-M_\infty^2}}$$



Problem 5. – During the testing of a two-dimensional, streamlined shape, it is found that sonic flow first occurs on the surface for $M_\infty = 0.70$. Calculate the pressure coefficient at this point and also the minimum pressure coefficient for this shape in incompressible flow.

The pressure coefficient in the point in which the sonic flow occurs is

$$C_{p\,M=1} = \frac{2}{\gamma M_{\infty crit}^2}\left[\frac{\left(1+\frac{\gamma-1}{2}M_{\infty crit}^2\right)^{\gamma/(\gamma-1)}}{\left(1+\frac{\gamma-1}{2}\right)^{\gamma/(\gamma-1)}}-1\right]$$

$$= \frac{2}{1.4\;0.7^2}\left[\frac{\left(1+\frac{1.4-1}{2}0.7^2\right)^{1.4/(1.4-1)}}{\left(1+\frac{1.4-1}{2}\right)^{1.4/(1.4-1)}}-1\right] = -0.779$$

The minimum pressure coefficient in incompressible flow is calculated from the Prandtl Glauert rule

$$C_{p\,M_\infty=0} = C_{p\,M_\infty}\sqrt{1-M_\infty^2} = -0.779\,\sqrt{1-0.7^2} = -0.556$$

Problem 6. – Two-dimensional subsonic linearized potential flow takes place between two wavy walls as shown Figure P13.6. Solve for ϕ_p and determine the pressure distribution along the centerline.

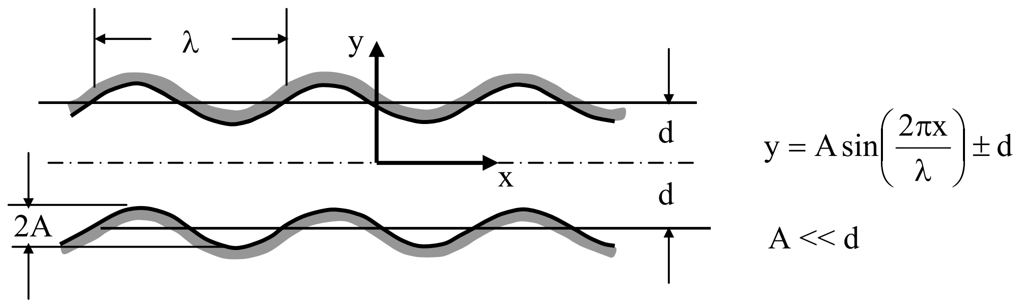


Figure P13.6

The potential equation for two-dimensional compressible flow is

$$\frac{\partial^2 \phi_p}{\partial x^2}\left(1 - M_\infty^2\right) + \frac{\partial^2 \phi_p}{\partial y^2} = 0$$

For subsonic flow, the solution of this equation can be obtained using the method of separation of variables

$$\phi_p = \left(c_1 \cos kx + c_2 \sin kx\right)\left(c_3 e^{\sqrt{1-M_\infty^2}\,ky} + c_4 e^{-\sqrt{1-M_\infty^2}\,ky}\right)$$

The constants c_1, c_2, c_3, c_4 can be determined from the boundary conditions. For $y = d$, the boundary condition is

$$\left(\frac{dy}{dx}\right)_b = \frac{v_p\left(x_b, d\right)}{U_\infty}$$

where

$$v_p\left(x_b, d\right) = \frac{\partial \phi_p}{\partial y}$$

Introducing the relationships for y_b and for ϕ_p into the boundary condition we receive a condition between the constants c_1, c_2, c_3, c_4

$$\left(c_1 \cos kx + c_2 \sin kx\right)\sqrt{1-M_\infty^2}\,k\left(c_3 e^{\sqrt{1-M_\infty^2}\ kd} - c_4 e^{-\sqrt{1-M_\infty^2}\ kd}\right) = \frac{2\pi U_\infty A}{\lambda}\cos\left(\frac{2\pi}{\lambda}x\right)$$

Similarly, for $y = -d$ we have

$$\left(\frac{dy}{dx}\right)_b = \frac{v_p\left(x_b, -d\right)}{U_\infty}$$

or

$$\left(c_1 \cos kx + c_2 \sin kx\right)\sqrt{1-M_\infty^2}\,k\left(c_3 e^{-\sqrt{1-M_\infty^2}\ kd} - c_4 e^{\sqrt{1-M_\infty^2}\ kd}\right) = \frac{2\pi U_\infty A}{\lambda}\cos\left(\frac{2\pi}{\lambda}x\right)$$

From the two relations between the constants c_1, c_2, c_3, c_4 we obtain

$$k = \frac{2\pi}{\lambda}$$
$$c_2 = 0$$
$$c_1 c_3 = \frac{U_\infty A}{\sqrt{1-M_\infty^2}}$$

With these relations, the potential function ϕ_p is

$$\phi_p = \frac{U_\infty A}{\sqrt{1-M_\infty^2}} \frac{\left(e^{\frac{2\pi}{\lambda}\sqrt{1-M_\infty^2}\ y} - e^{-\frac{2\pi}{\lambda}\sqrt{1-M_\infty^2}\ y}\right)}{\left(e^{\frac{2\pi}{\lambda}\sqrt{1-M_\infty^2}\ d} + e^{-\frac{2\pi}{\lambda}\sqrt{1-M_\infty^2}\ d}\right)} \cos\left(\frac{2\pi}{\lambda}x\right)$$

Along the centerline we have

$$u_{p\ y=0} = \left(\frac{\partial \phi_p}{\partial x}\right)_{y=0} = 0$$

which implies

$$C_{p\ y=0} = -\frac{2u_p}{U_\infty} = 0$$

Because

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

we obtain that along the centerline the pressure is constant

$$p = p_\infty$$

***Problem** 7*. – Consider two-dimensional, supersonic, linearized flow under a wavy wall, as shown in Figure P13.7. Solve for the velocity potential of the flow and pressure coefficient along the wall. Derive an expression for the lift and drag per wave length.

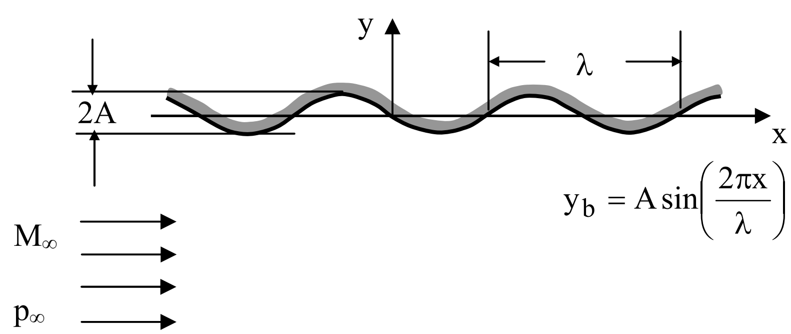


Figure P13.7

The general solution of the linearized potential equation two-dimensional, supersonic flow under a wall is

$$\phi_p = f\left(x + \sqrt{M_\infty^2 - 1}\ y\right)$$

The boundary condition is

$$\left(\frac{dy}{dx}\right)_b = \left(\frac{v}{U_\infty}\right)_{y=0} = \frac{1}{U_\infty}\left(\frac{\partial \phi_p}{\partial y}\right)_{y=0}$$

For the wavy wall, this boundary condition becomes

$$\frac{2\pi A}{\lambda}\cos\left(\frac{2\pi}{\lambda}x\right)=\frac{1}{U_\infty}\left[\frac{df}{d\left(x+\sqrt{M_\infty^2-1}\,y\right)}\right]_{y=0}\sqrt{M_\infty^2-1}$$

$$\left[\frac{df}{d\left(x+\sqrt{M_\infty^2-1}\,y\right)}\right]_{y=0}=\frac{U_\infty}{\sqrt{M_\infty^2-1}}\frac{2\pi A}{\lambda}\cos\left(\frac{2\pi}{\lambda}x\right)$$

or, any y,

$$\frac{df}{d\left(x+\sqrt{M_\infty^2-1}y\right)}=\frac{U_\infty}{\sqrt{M_\infty^2-1}}\frac{2\pi A}{\lambda}\cos\left(\frac{2\pi\left(x+\sqrt{M_\infty^2-1}\,y\right)}{\lambda}\right)$$

Integrating, we obtain

$$f=\frac{U_\infty A}{\sqrt{M_\infty^2-1}}\sin\left(\frac{2\pi\left(x+\sqrt{M_\infty^2-1}\,y\right)}{\lambda}\right)+\text{constant}=\phi_p$$

The perturbation velocity u_p is

$$u_p=\frac{\partial\phi_p}{\partial x}=\frac{U_\infty A}{\sqrt{M_\infty^2-1}}\frac{2\pi}{\lambda}\cos\left(\frac{2\pi\left(x+\sqrt{M_\infty^2-1}\,y\right)}{\lambda}\right)$$

With u_p we can compute the pressure coefficient C_p along the wall

$$C_p=\left(-\frac{2u_p}{U_\infty}\right)_{y=0}=-\frac{4\pi A}{\lambda\sqrt{M_\infty^2-1}}\cos\left(\frac{2\pi}{\lambda}x\right)$$

The differential lift dL is given by

$$dL=C_p\frac{1}{2}\gamma p_\infty M_\infty^2dx+p_\infty dx$$

where lift is defined to be positive upward.

Integrating from 0 to λ, we obtain the lift force per wave length

$$L = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^\lambda C_p dx + \int_0^\lambda p_\infty dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{4\pi A}{\lambda\sqrt{M_\infty^2 - 1}} \right) \int_0^\lambda \cos\left(\frac{2\pi}{\lambda} x \right) dx + p_\infty \int_0^\lambda dx = p_\infty \lambda$$

The differential drag is

$$dD = -C_p \frac{1}{2}\gamma p_\infty M_\infty^2 \frac{dy}{dx} dx - p_\infty \frac{dy}{dx} dx$$

where drag is positive in the flow direction. Integrating from 0 to λ, we have the drag per wave length

$$D = -\frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^\lambda \left(C_p \frac{dy}{dx} \right) dx - \int_0^\lambda \left(p_\infty \frac{dy}{dx} \right) dx =$$

$$-\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{4\pi A}{\lambda\sqrt{M_\infty^2 - 1}} \right) \left(\frac{2\pi A}{\lambda} \right) \int_0^\lambda \cos^2\left(\frac{2\pi}{\lambda} x \right) dx - p_\infty \left(\frac{2\pi A}{\lambda} \right) \int_0^\lambda \cos\left(\frac{2\pi}{\lambda} x \right) dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{4\pi A}{\lambda\sqrt{M_\infty^2 - 1}} \right) \left(\frac{2\pi A}{\lambda} \right) \frac{\lambda}{2} =$$

$$\frac{2\pi^2 A^2}{\lambda} \gamma p_\infty \frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}$$

Problem 8. – A wing has the shape of a sine wave, as shown in Figure P13.8. Compute the lift and drag for supersonic flow. Assume linearized, two-dimensional flow above and below the foil.

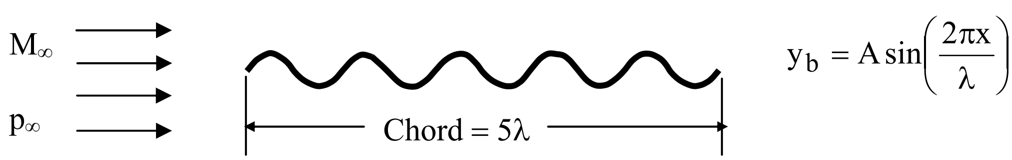


Figure P13.8

For a wing with a shape of a sine wave we have

$$\left(\frac{dy_b}{dx}\right)_L = \left(\frac{dy_b}{dx}\right)_u = \frac{2\pi A}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)$$

Introducing these relations in the expression of the lift for an airfoil, we have

$$L = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \left(-\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left\{\left(\frac{dy_b}{dx}\right)_L + \left(\frac{dy_b}{dx}\right)_u\right\}dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^{5\lambda} \left(-\frac{2}{\sqrt{M_\infty^2 - 1}}\right) 2\frac{2\pi A}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\frac{4\pi A}{\lambda}\int_0^{5\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)dx = 0$$

For the drag, in the same manner we have

$$D = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \frac{2}{\sqrt{M_\infty^2 - 1}}\left[\left(\frac{dy_b}{dx}\right)_u^2 + \left(\frac{dy_b}{dx}\right)_l^2\right]dx + p_\infty \int_0^c \left[\left(\frac{dy_b}{dx}\right)_u - \left(\frac{dy_b}{dx}\right)_L\right]dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \frac{2}{\sqrt{M_\infty^2 - 1}}\int_0^{5\lambda} 2\left[\frac{2\pi A}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2 dx =$$

$$\gamma p_\infty \frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{8\pi^2 A^2}{\lambda^2}\int_0^{5\lambda}\left[\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2 dx =$$

$$\gamma p_\infty \frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{8\pi^2 A^2}{\lambda^2}\frac{5\lambda}{2} =$$

$$\gamma p_\infty \frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}\frac{20\pi^2 A^2}{\lambda}$$

Problem 9. – Using thin airfoil theory, find C_L and C_D for a two-dimensional, flat plate airfoil with deflected flap in supersonic flow of Mach number M_∞. Plot C_L versus α for various δ for F=0.25 (Figure P13.9).

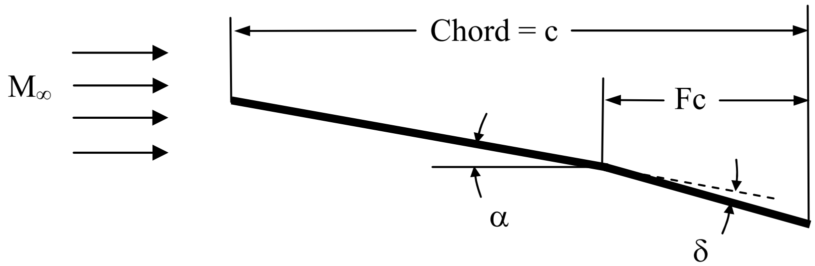


Figure P13.9

For the airfoil from Figure P13.9 we can distinguish two different regions:

$$0 \le x \le (1-F)c \qquad \left(\frac{dy}{dx}\right)_l = \left(\frac{dy}{dx}\right)_u = -\alpha$$

$$(1-F)c \le x \le c \qquad \left(\frac{dy}{dx}\right)_l = \left(\frac{dy}{dx}\right)_u = -(\alpha+\delta)$$

We substitute these relations in the general expression of the lift for an airfoil

$$L = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \left(-\frac{2}{\sqrt{M_\infty^2-1}}\right)\left\{\left(\frac{dy}{dx}\right)_l + \left(\frac{dy}{dx}\right)_u\right\}dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{2}{\sqrt{M_\infty^2-1}}\right)\int_0^c 2\left(\frac{dy}{dx}\right)_l dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{4}{\sqrt{M_\infty^2-1}}\right)\left\{\int_0^{(1-F)c}(-\alpha)dx + \int_{(1-F)c}^c[-(\alpha+\delta)]dx\right\} =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(\frac{4}{\sqrt{M_\infty^2-1}}\right)[\alpha(1-F)c + (\alpha+\delta)Fc] =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 c\left(\frac{4}{\sqrt{M_\infty^2-1}}\right)(\alpha+\delta F)$$

Therefore, the lift coefficient C_L is

$$C_L = \frac{L}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} = \frac{4}{\sqrt{M_\infty^2-1}}(\alpha+\delta F)$$

Particularly, for F=0.25 C_L is

$$C_L = \frac{4}{\sqrt{M_\infty^2 - 1}}(\alpha + 0.25\delta)$$

For the drag we get

$$D = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \left(\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left[\left(\frac{dy}{dx}\right)_u^2 + \left(\frac{dy}{dx}\right)_l^2\right]dx + p_\infty \int_0^c \left[\left(\frac{dy}{dx}\right)_u - \left(\frac{dy}{dx}\right)_l\right]dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^c 2\left(\frac{dy}{dx}\right)_l^2 dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\int_0^{(1-F)c} \alpha^2 dx + \int_{(1-F)c}^c (\alpha + \delta)^2 dx\right) =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \frac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 (1-F)c + (\alpha + \delta)^2 Fc\right] =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 c \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + \delta^2 F + 2\alpha\delta F\right)$$

and the drag coefficient C_D is

$$C_D = \frac{D}{\frac{1}{2}\gamma p_\infty M_\infty^2 c} = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + \delta^2 F + 2\alpha\delta F\right)$$

Consequently for F=0.25 C_D is

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + 0.25\delta^2 + 0.5\alpha\delta\right)$$

Problem 10. – Consider uniform supersonic flow over a wall in which there exists a bump, as shown in Figure P13.10. Assuming linearized, two-dimensional potential flow, calculate the vertical and horizontal components of the force on the bump. Assume M_∞ = 2.0, with p_∞ = 50 kPa.

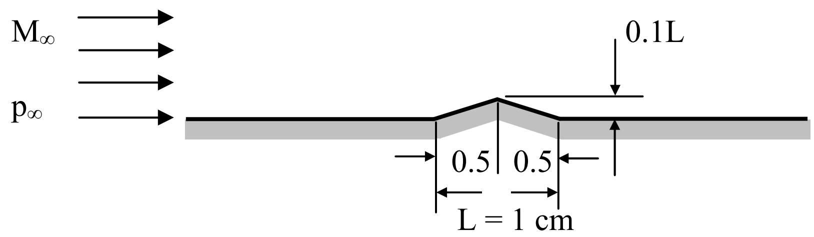


Figure P13.10

For the wall from Figure P13.10 we can distinct two different regions where dy/dx is not zero

$$0 \le x \le 0.5L \quad \frac{dy}{dx} = \frac{0.1L}{\frac{L}{2}} = 0.2$$

$$0.5L \le x \le L \quad \frac{dy}{dx} = -\frac{0.1L}{\frac{L}{2}} = -0.2$$

Using these relations in the general expression for the lift we get

$$L = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^L \left(-\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left(\frac{dy}{dx}\right)dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(-\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left[\int_0^{0.5L} (0.2)dx + \int_{0.5L}^{L} (-0.2)dx\right] = 0$$

For the drag we can write

$$D = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \left(\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left(\frac{dy}{dx}\right)^2 dx + p_\infty \int_0^c \left(\frac{dy}{dx}\right)dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \frac{2}{\sqrt{M_\infty^2 - 1}}\left[\int_0^{0.5L} (0.2)^2 dx + \int_{0.5L}^{L} (-0.2)^2 dx\right] =$$

$$\gamma p_\infty \frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}} 0.04L = (1.4)\left(50 \cdot 10^3\right)\frac{2^2}{\sqrt{2^2 - 1}}(0.04)\left(1 \cdot 10^{-2}\right) = 64.66\text{N}$$

Problem 11. – A supersonic airfoil consists of a circular arc, as shown in Figure P13.11. Compute the lift and drag coefficients of the foil versus angle of attack.

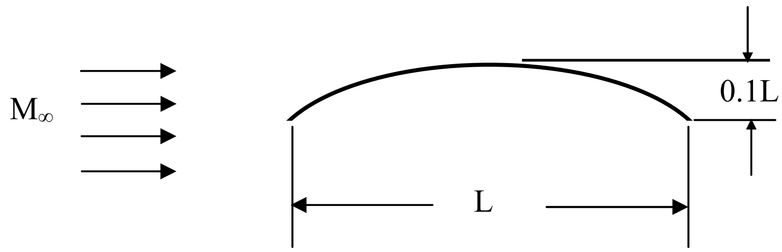


Figure P13.1l

The lift coefficient is

$$C_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

The airfoil with a shape of a circular arc has camber but no thickness, with

$$\overline{\frac{dC}{dx}} = \frac{0.1L}{\frac{L}{2}} = 0.2$$

Thus, the drag coefficient is

$$C_D = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 + \overline{\left(\frac{dC}{dx}\right)^2}\right] = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 + 0.2^2\right] =$$
$$\frac{4}{\sqrt{M_\infty^2 - 1}}\left[\alpha^2 + 0.04\right]$$

Problem 12. – For the airfoil shown in Figure P13.12, determine C_L and C_D versus angle of attack in supersonic flow.

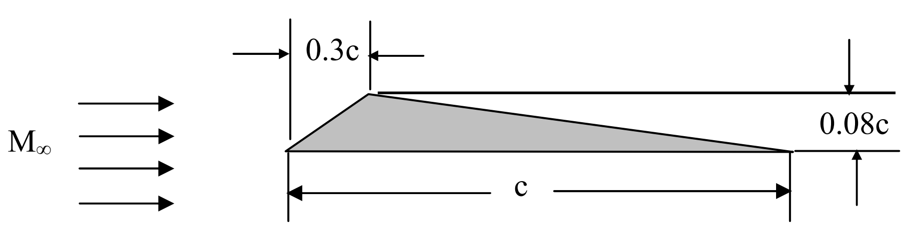


Figure P13.12

The lift coefficient is

$$C_L = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

For the upper surface of the airfoil, dy/dx can have two distinctive values

$$\left(\frac{dy}{dx}\right)_u = \frac{0.08c}{0.3c} = \frac{0.8}{3} \quad \text{for } 0 \le x \le 0.3c$$

and

$$\left(\frac{dy}{dx}\right)_u = \frac{0.08c}{0.7c} = \frac{0.8}{7} \quad \text{for } 0.3c \le x \le c$$

For the lower surface of the airfoil

$$\left(\frac{dy}{dx}\right)_l = 0$$

For zero angle of attack, the drag is

$$D = \frac{1}{2}\gamma p_\infty M_\infty^2 \int_0^c \left(\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left[\left(\frac{dy}{dx}\right)_u^2 + \left(\frac{dy}{dx}\right)_l^2\right]dx + p_\infty \int_0^c \left[\left(\frac{dy}{dx}\right)_u - \left(\frac{dy}{dx}\right)_l\right]dx =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 \left(\frac{2}{\sqrt{M_\infty^2 - 1}}\right)\left[\int_0^{0.3c}\left(\frac{0.8}{3}\right)^2 dx + \int_{0.3c}^{c}\left(-\frac{0.8}{7}\right)^2 dx\right] =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 c\left(\frac{2}{\sqrt{M_\infty^2 - 1}}\right)0.03047 =$$

$$\frac{1}{2}\gamma p_\infty M_\infty^2 c\frac{0.06095}{\sqrt{M_\infty^2 - 1}}$$

Consequently, for zero angle of attack, the drag coefficient is

$$C_D = \frac{0.06095}{\sqrt{M_\infty^2 - 1}}$$

For an angle of attack α, we have an additional term

$$C_D = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} + \frac{0.06095}{\sqrt{M_\infty^2 - 1}}$$

Chapter Fourteen

CHARACTERISTICS

Problem 1. – Use the Method of Indeterminate Derivatives to obtain equations of the characteristics for the following equation in the hodograph plane,

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\Phi}{\partial v^2}+2\frac{uv}{a^2}\frac{\partial^2\Phi}{\partial u\partial v}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\Phi}{\partial u^2}=0$$

where the function, $\Phi(u,v)$, in the hodograph plane is related to the velocity potential, $\phi(x,y)$, by

$$\Phi = xu + yv - \phi$$

so that

$$\frac{\partial\Phi}{\partial u}=x \ \text{ and } \ \frac{\partial\Phi}{\partial v}=y$$

(see Problem 10 in Chapter 12).

To begin write the total derivatives of $\partial\Phi/\partial u$ and $\partial\Phi/\partial v$

$$du\frac{\partial^2\Phi}{\partial u^2}+dv\frac{\partial^2\Phi}{\partial v\partial u}=d\left(\frac{\partial\Phi}{\partial u}\right)=dx$$

$$du\frac{\partial^2\Phi}{\partial u\partial v}+dv\frac{\partial^2\Phi}{\partial v^2}=d\left(\frac{\partial\Phi}{\partial v}\right)=dy$$

Next rewrite

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\Phi}{\partial v^2}+2\frac{uv}{a^2}\frac{\partial^2\Phi}{\partial u\partial v}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial^2\Phi}{\partial u^2}=0$$

in the following form

$$C_1\frac{\partial^2\Phi}{\partial u^2}+2C_2\frac{\partial^2\Phi}{\partial u\partial v}+C_3\frac{\partial^2\Phi}{\partial v^2}=0$$

Note two of the terms have changed places and the coefficients are

$$C_1 = \left(1 - \frac{v^2}{a^2}\right)$$

$$C_2 = \frac{uv}{a^2}$$

$$C_3 = \left(1 - \frac{u^2}{a^2}\right)$$

Since $\partial^2\Phi/\partial u\partial v = \partial^2\Phi/\partial v\partial u$, the above provide three equations in terms of the three unknown second derivatives. Solving for any of the derivatives (here $\partial^2\Phi/\partial u^2$ was selected) using Cramer's Rule yields

$$\frac{\partial^2\Phi}{\partial u^2} = \frac{|N|}{|D|}$$

where

$$|N| = \begin{vmatrix} dx & dv & 0 \\ dy & du & dv \\ 0 & 2C_2 & C_3 \end{vmatrix} \text{ and } |D| = \begin{vmatrix} du & dv & 0 \\ 0 & du & dv \\ C_1 & 2C_2 & C_3 \end{vmatrix}$$

Setting the determinant D to zero gives the equation of the characteristic in the hodograph plane, i.e.,

$$|D| = \begin{vmatrix} du & dv & 0 \\ 0 & du & dv \\ C_1 & 2C_2 & C_3 \end{vmatrix} = du\left(C_3 du - 2C_2 dv\right) - dv\left(0 - C_1 dv\right) = 0$$

which produces the following quadratic

$$C_1\left(\frac{dv}{du}\right)^2 - 2C_2\left(\frac{dv}{du}\right) + C_3 = 0$$

Solving this gives

$$\frac{dv}{du} = \frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1}$$

Substitution brings Eq.(14.21)

$$\frac{dv}{du} = \frac{uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - v^2}$$

The equation for the information that is carried on the characteristics is obtained by equating the determinant N to zero, i.e.,

$$|N| = \begin{vmatrix} dx & dv & 0 \\ dy & du & dv \\ 0 & 2C_2 & C_3 \end{vmatrix} = dx\left(C_3 du - 2C_2 dv\right) - dy\left(C_3 dv - 0\right) = 0$$

which produces

$$\frac{dy}{dx} = \frac{du}{dv} - 2\frac{C_2}{C_3} = \frac{C_1}{C_2 \pm \sqrt{C_2^2 - C_1 C_3}} - \frac{2C_2}{C_3}$$

$$= \frac{C_1}{C_2 \pm \sqrt{C_2^2 - C_1 C_3}}\left(\frac{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}\right) - \frac{2C_2}{C_3}$$

$$= \frac{C_1\left(-C_2 \pm \sqrt{C_2^2 - C_1 C_3}\right)}{-C_2^2 + C_2^2 - C_1 C_3} - \frac{2C_2}{C_3}$$

$$= \frac{-\left(-C_2 \pm \sqrt{C_2^2 - C_1 C_3}\right) - 2C_2}{C_3}$$

$$= \frac{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}{C_3}$$

Substitution brings Eq.(14.15)

$$\frac{dy}{dx} = \frac{-uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - u^2}$$

Problem 2. – Use the Method of Indeterminate Derivatives to obtain the equation for the slope of the characteristics in the hodograph plane in terms of the flow speed V and the flow angle α. Develop the equation starting with the potential equation in the hodograph plane, i.e., Eq.(12.55)

$$V^2\left(1 - \frac{V^2}{a^2}\right)\frac{\partial^2\phi}{\partial V^2} + V\left(1 + \gamma\frac{V^4}{a^4}\right)\frac{\partial\phi}{\partial V} + \left(1 - \frac{V^2}{a^2}\right)^2\frac{\partial^2\phi}{\partial\alpha^2} = 0$$

Write the total derivatives of $\partial\phi/\partial V$ and $\partial\phi/\partial\alpha$

$$dV\frac{\partial^2\phi}{\partial V^2} + d\alpha\frac{\partial^2\phi}{\partial\alpha\partial V} = d\left(\frac{\partial\phi}{\partial V}\right)$$

$$dV\frac{\partial^2\phi}{\partial V\partial\alpha} + d\alpha\frac{\partial^2\phi}{\partial\alpha^2} = d\left(\frac{\partial\phi}{\partial\alpha}\right)$$

Next rewrite the potential equation as

$$C_1\frac{\partial^2\phi}{\partial V^2} + C_3\frac{\partial^2\phi}{\partial\alpha^2} + C_4\frac{\partial\phi}{\partial V} = 0$$

The coefficients are

$$\begin{aligned}
C_1 &= V^2\left(1-\frac{V^2}{a^2}\right)\\
C_2 &= 0\\
C_3 &= \left(1-\frac{V^2}{a^2}\right)^2\\
C_4 &= V\left(1+\gamma\frac{V^4}{a^4}\right)
\end{aligned}$$

The above provide three equations in terms of the three unknown second derivatives. Solving for any of the derivatives (here $\partial^2\phi/\partial V^2$ was selected) using Cramer's Rule yields

$$\frac{\partial^2\phi}{\partial V^2} = \frac{|N|}{|D|}$$

where

$$|N| = \begin{vmatrix} d(\phi_V) & d\alpha & 0\\ d(\phi_\alpha) & dV & d\alpha\\ 0 & 0 & C_3 \end{vmatrix} \text{ and } |D| = \begin{vmatrix} dV & d\alpha & 0\\ 0 & dV & d\alpha\\ C_1 & 0 & C_3 \end{vmatrix}$$

Setting the determinant D to zero gives the desired equation of the characteristic in the hodograph plane, i.e.,

$$|D| = \begin{vmatrix} dV & d\alpha & 0 \\ 0 & dV & d\alpha \\ C_1 & 0 & C_3 \end{vmatrix} = dV(C_3 dV - 0) - d\alpha(0 - C_1 d\alpha) = 0$$

which produces the following

$$\left(\frac{dV}{d\alpha}\right)^2 = -\frac{C_1}{C_3} = -\frac{V^2\left(1-\frac{V^2}{a^2}\right)}{\left(1-\frac{V^2}{a^2}\right)^2}$$

$$= \frac{V^2}{\left(\frac{V^2}{a^2}-1\right)} = \frac{V^2}{\left(M^2-1\right)}$$

or

$$\frac{1}{V}\frac{dV}{d\alpha} = \pm\frac{1}{\sqrt{M^2-1}} = \pm\tan\mu$$

which is Eq.(14.52).

Problem 3. – Use the Method of Linear Combination to obtain equations of the characteristics for the following set of equations in the hodograph plane,

$$\left(a^2-u^2\right)\frac{\partial y}{\partial v} + uv\left(\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\right) + \left(a^2-v^2\right)\frac{\partial x}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0$$

Rewrite the pair of first-order partial differential equations as

$$C_1\frac{\partial x}{\partial u} + C_2\left(\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\right) + C_3\frac{\partial y}{\partial v} = 0$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0$$

Note the end terms of the potential equation have been interchanged and the coefficients are

$$
\begin{aligned}
C_1 &= \left(a^2 - v^2\right) \\
C_2 &= uv \\
C_3 &= \left(a^2 - u^2\right)
\end{aligned}
$$

Now multiply the first equation by an unknown parameter σ_1, the second by σ_2 and add the results to get

$$\sigma_1\left[C_1\frac{\partial x}{\partial u} + C_2\left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\right) + C_3\frac{\partial y}{\partial v}\right] + \sigma_2\left(\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u}\right) = 0$$

Grouping like derivatives gives

$$\left[C_1\sigma_1\frac{\partial x}{\partial u} + \left(C_2\sigma_1 + \sigma_2\right)\frac{\partial x}{\partial v}\right] + \left[\left(C_2\sigma_1 - \sigma_2\right)\frac{\partial y}{\partial u} + C_3\sigma_1\frac{\partial y}{\partial v}\right] = 0$$

Or

$$C_1\sigma_1\left[\frac{\partial x}{\partial u} + \frac{\left(C_2\sigma_1 + \sigma_2\right)}{C_1\sigma_1}\frac{\partial x}{\partial v}\right] + \left(C_2\sigma_1 - \sigma_2\right)\left[\frac{\partial y}{\partial u} + \frac{C_3\sigma_1}{\left(C_2\sigma_1 - \sigma_2\right)}\frac{\partial y}{\partial v}\right] = 0$$

Compare the group of terms within the square brackets to the following total derivatives

$$\frac{\partial x}{\partial u} + \frac{dv}{du}\frac{\partial y}{\partial v} = \frac{dx}{du}$$

$$\frac{\partial y}{\partial u} + \frac{dv}{du}\frac{\partial y}{\partial v} = \frac{dy}{du}$$

From this comparison we may write the slope, dv/du, denoted as λ, as

$$\frac{dv}{du} = \lambda = \frac{\left(C_2\sigma_1 + \sigma_2\right)}{C_1\sigma_1} = \frac{C_3\sigma_1}{\left(C_2\sigma_1 - \sigma_2\right)}$$

Expanding this pair of equations produces two equations for σ_1 and σ_2

$$\left(C_1\lambda - C_2\right)\sigma_1 - \sigma_2 = 0$$

$$\left(C_2\lambda - C_3\right)\sigma_1 - \lambda\sigma_2 = 0$$

A unique solution for σ_1 and σ_2 will be obtained if and only if the determinant of the coefficients vanishes, i.e.,

$$\begin{vmatrix} (C_1\lambda - C_2) & -1 \\ (C_2\lambda - C_3) & -\lambda \end{vmatrix} = 0$$

Expanding and rearranging the result produces the quadratic equation,

$$C_1\lambda^2 - 2C_2\lambda + C_3 = 0$$

Solution of this expression yields

$$\lambda = \frac{dv}{du} = \frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1} = \frac{uv \pm \sqrt{a^2(u^2 + v^2) - a^4}}{a^2 - v^2}$$

which was previously obtained as Eq.(14.21).

To derive the compatibility equation, incorporate the total derivative equations into the combined equation and obtain

$$C_1\sigma_1 dx + (C_2\sigma_1 - \sigma_2)dy = 0$$

or

$$C_1 dx + \left(C_2 - \frac{\sigma_2}{\sigma_1}\right)dy = 0$$

The relationship between σ_1 and σ_2 is obtained from use of either

$$(C_1\lambda - C_2)\sigma_1 - \sigma_2 = 0$$

$$(C_2\lambda - C_3)\sigma_1 - \lambda\sigma_2 = 0$$

Using the first of these and the expression for λ produces

$$\frac{\sigma_2}{\sigma_1} = C_1\lambda - C_2 = C_1\left(\frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1}\right) - C_2 = \pm\sqrt{C_2^2 - C_1C_3}$$

Hence,

$$C_1 dx + \left(C_2 \mp \sqrt{C_2^2 - C_1C_3}\right)dy = 0$$

So

$$\frac{dy}{dx} = \frac{C_1}{-C_2 \pm \sqrt{C_2^2 - C_1C_3}} = \frac{C_1}{-C_2 \pm \sqrt{C_2^2 - C_1C_3}}\,\frac{-C_2 \mp \sqrt{C_2^2 - C_1C_3}}{-C_2 \mp \sqrt{C_2^2 - C_1C_3}}$$

$$\frac{dy}{dx} = \frac{C_1\left(-C_2 \mp \sqrt{C_2^2 - C_1C_3}\right)}{C_2^2 - \left(C_2^2 - C_1C_3\right)} = \frac{-C_2 \mp \sqrt{C_2^2 - C_1C_3}}{C_3}$$

Substitution brings Eq.(14.15)

$$\frac{dy}{dx} = \frac{-uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - u^2}$$

Problem 4. – Resolve problem 3 using Eigenanalysis

The pair of equations

$$C_1\frac{\partial x}{\partial u} + C_2\left(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\right) + C_3\frac{\partial y}{\partial v} = 0$$

$$\frac{\partial x}{\partial v} - \frac{\partial y}{\partial u} = 0$$

can be written in vector matrix form as

$$\begin{bmatrix} C_1 & C_2 \\ 0 & -1 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial u} + \begin{bmatrix} C_2 & C_3 \\ 1 & 0 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial v} = 0$$

where the dependent column vector is $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. Defining the coefficient matrices as

$$\mathbf{A} = \begin{bmatrix} C_1 & C_2 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} C_2 & C_3 \\ 1 & 0 \end{bmatrix}$$

the above equation may be written as

$$\mathbf{A}\frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{w}}{\partial y} = 0$$

So

$$\frac{\partial \mathbf{w}}{\partial x} + \mathbf{A}^{-1}\mathbf{B}\frac{\partial \mathbf{w}}{\partial y} = \frac{\partial \mathbf{w}}{\partial x} + \mathbf{C}\frac{\partial \mathbf{w}}{\partial y} = 0$$

The inverse matrix of **A** is

$$\mathbf{A}^{-1} = \frac{\text{adj}\,\mathbf{A}}{\det \mathbf{A}} = \frac{1}{\left(-C_1 - 0\right)}\begin{bmatrix} -1 & -C_2 \\ 0 & C_1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{C_1} & \dfrac{C_2}{C_1} \\ 0 & -1 \end{bmatrix}$$

Multiplying $\mathbf{A}^{-1}$ times **B** gives **C**

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \dfrac{1}{C_1} & \dfrac{C_2}{C_1} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} C_2 & C_3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{2C_2}{C_1} & \dfrac{C_3}{C_1} \\ -1 & 0 \end{bmatrix}$$

The characteristic directions are obtained by determining the *eigenvalues* of matrix **C**,

$$\left|\mathbf{C} - \lambda\mathbf{I}\right| = 0$$

that is

$$\begin{vmatrix} \dfrac{2C_2}{C_1} - \lambda & \dfrac{C_3}{C_1} \\ -1 & -\lambda \end{vmatrix} = 0$$

Expanding the determinant yields the same quadratic expression as obtained by the two previous methods problems 3 and 4

$$-\left(\frac{2C_2}{C_1} - \lambda\right)\lambda + \frac{C_3}{C_1} = 0$$

$$C_1\lambda^2 - 2C_2\lambda + C_3 = 0$$

Solution of this brings

$$\lambda = \frac{dv}{du} = \frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1} = \frac{uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - v^2}$$

To derive the compatibility equation, begin with

$$\frac{\partial \mathbf{w}}{\partial x} + \mathbf{C}\frac{\partial \mathbf{w}}{\partial y} = 0$$

The left eigenvectors corresponding to eigenvalue, λ_i, of matrix **C** are determined from

$$\mathbf{l}_i^T\mathbf{C} = \lambda_i\mathbf{l}_i^T$$

or

$$\mathbf{l}_i^T\left(\mathbf{C} - \lambda_i\mathbf{I}\right) = 0$$

The characteristic variables are defined by

$$d\boldsymbol{\Gamma} = \mathbf{X}^{-1} d\mathbf{w} = 0$$

where

$$\mathbf{X}^{-1} = \mathbf{L} = \begin{bmatrix} \mathbf{l}_{\mathrm{I}} & \mathbf{l}_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} l_{1\mathrm{I}} & l_{1\mathrm{II}} \\ l_{2\mathrm{I}} & l_{2\mathrm{II}} \end{bmatrix}$$

To derive the compatibility equation, the left eigenvectors must first be determined. Rather than obtaining results for each characteristic (I and II), the following applies to characteristics of either family

$$\mathbf{l}^{\mathrm{T}} \left(\mathbf{C} - \lambda \mathbf{I} \right) = 0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix} \begin{bmatrix} \dfrac{2C_2}{C_1} - \lambda & \dfrac{C_3}{C_1} \\ -1 & -\lambda \end{bmatrix} = 0$$

Expanding gives two equations

$$\left(2\frac{C_2}{C_1} - \lambda \right) l_1 - l_2 = 0$$

$$\frac{C_3}{C_1} l_1 - \lambda l_2 = 0$$

Hence,

$$l_2 = \left(2\frac{C_2}{C_1} - \lambda \right) l_1$$

$$l_2 = \left(\frac{C_3}{\lambda C_1} \right) l_1$$

The group of coefficients on the right side of the above two equations are equal to each other as may be seen by examining the eigenvalue expansion. Therefore, the equations are not independent of each other. So we may arbitrarily assign a value to either l_1 or l_2 and then use the above to determine the remaining component. Let $l_1 = 1$, so that

$$l_2 = \left(2\frac{C_2}{C_1} - \lambda \right) = \left(\frac{C_3}{\lambda C_1} \right).$$

The compatibility equation for characteristics is

$$\mathbf{l}^{\mathrm{T}} d\mathbf{w} = 0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 1 & \left(2\frac{C_2}{C_1} - \lambda\right) \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} = dx + \left(2\frac{C_2}{C_1} - \lambda\right) dy = 0$$

Consequently,

$$\frac{dy}{dx} = \frac{1}{\lambda - 2\frac{C_2}{C_1}} = \frac{1}{\lambda} + \frac{2\frac{C_2}{C_1}}{\lambda\left(\lambda - 2\frac{C_2}{C_1}\right)}$$

And since $\left(2\frac{C_2}{C_1} - \lambda\right) = \left(\frac{C_3}{\lambda C_1}\right)$ the above becomes

$$\frac{dy}{dx} = \frac{1}{\lambda} + \frac{2\frac{C_2}{C_1}}{\lambda\left(-\frac{C_3}{\lambda C_1}\right)} = \frac{1}{\lambda} - \frac{2C_2}{C_3}$$

$$\frac{dy}{dx} = \frac{du}{dv} - 2\frac{C_2}{C_3} = \frac{C_1}{C_2 \pm \sqrt{C_2^2 - C_1 C_3}} - \frac{2C_2}{C_3}$$

$$= \frac{C_1}{C_2 \pm \sqrt{C_2^2 - C_1 C_3}}\left(\frac{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}\right) - \frac{2C_2}{C_3}$$

$$= \frac{C_1\left(-C_2 \pm \sqrt{C_2^2 - C_1 C_3}\right)}{-C_2^2 + C_2^2 - C_1 C_3} - \frac{2C_2}{C_3}$$

$$= \frac{-\left(-C_2 \pm \sqrt{C_2^2 - C_1 C_3}\right) - 2C_2}{C_3}$$

$$= \frac{-C_2 \pm \sqrt{C_2^2 - C_1 C_3}}{C_3}$$

Substitution brings

$$\frac{dy}{dx} = \frac{-uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - u^2}$$

Problem 5. –The continuity and momentum equations for one-dimensional unsteady flow are

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

(a) For an isentropic flow show that this pair can be written as

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}$$

(b) Define the Riemann variable, R, as

$$dR = a\frac{d\rho}{\rho}$$

and show that the pair of equations in part (a) become

$$\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial R}{\partial x} = 0$$

(c) Add and subtract the pair of equations in part (b) to obtain

$$\frac{\partial\left(u + R\right)}{\partial t} + \left(u + a\right)\frac{\partial\left(u + R\right)}{\partial x} = 0$$

$$\frac{\partial\left(u - R\right)}{\partial t} + \left(u - a\right)\frac{\partial\left(u - R\right)}{\partial x} = 0$$

(d) From the results of part (c) determine the slope of the characteristics, i.e., dx/dt, as well as the information that is propagated along the characteristics.

(e) For isentropic flow of a perfect gas, show that if R(0) = a(0) = 0, then

$$R = \frac{2}{\gamma - 1}a$$

(a) Since the flow is isentropic we have

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{dp}{d\rho}$$

or

$$dp = a^2 d\rho$$

So

$$\frac{\partial p}{\partial x} = a^2 \frac{\partial \rho}{\partial x}$$

Hence, the original pair

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x}$$

become

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}$$

(b) The Riemann variable R is defined by

$$dR = a\frac{d\rho}{\rho}$$

So

$$\frac{\partial R}{\partial t} = \frac{a}{\rho}\frac{\partial \rho}{\partial t}$$

$$\frac{\partial R}{\partial x} = \frac{a}{\rho}\frac{\partial \rho}{\partial x}$$

Use these to replace the density derivatives in the pair

$$\frac{\partial \rho}{\partial t}+\rho \frac{\partial u}{\partial x}+u \frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}=-\frac{a^2}{\rho} \frac{\partial \rho}{\partial x}$$

To obtain

$$\frac{\rho}{a} \frac{\partial R}{\partial t}+\rho \frac{\partial u}{\partial x}+\frac{\rho u}{a} \frac{\partial R}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+\frac{a^2}{\rho} \frac{\rho}{a} \frac{\partial R}{\partial x}=0$$

Performing the cancellation of terms gives

$$\frac{\partial R}{\partial t}+u \frac{\partial R}{\partial x}+a \frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+a \frac{\partial R}{\partial x}=0$$

(c) Add the pair in (b) to get

$$\frac{\partial u}{\partial t}+\frac{\partial R}{\partial t}+u \frac{\partial u}{\partial x}+u \frac{\partial R}{\partial x}+a \frac{\partial u}{\partial x}+a \frac{\partial R}{\partial x}=$$

$$\frac{\partial (u+R)}{\partial t}++(u+a) \frac{\partial (u+R)}{\partial x}=0$$

Subtraction produces

$$\frac{\partial u}{\partial t}-\frac{\partial R}{\partial t}+u \frac{\partial u}{\partial x}-u \frac{\partial R}{\partial x}+a \frac{\partial u}{\partial x}-a \frac{\partial R}{\partial x}=$$

$$\frac{\partial (u-R)}{\partial t}++(u-a) \frac{\partial (u-R)}{\partial x}=0$$

(d) The pair of pde in (c) can be rewritten collectively as

$$\frac{\partial(u \pm R)}{\partial t} + (u \pm a)\frac{\partial(u \pm R)}{\partial x} = 0$$

Contrast this to the total derivative of $u \pm R$, i.e.,

$$\frac{\partial(u \pm R)}{\partial t}dt + \frac{\partial(u \pm R)}{\partial x}dx = d(u \pm R)$$

Rather to

$$\frac{\partial(u \pm R)}{\partial t} + \frac{dx}{dt}\frac{\partial(u \pm R)}{\partial x} = \frac{d(u \pm R)}{dt}$$

And we observe that the quantity $u \pm R$ remains constant along a line whose slope is

$$\frac{dx}{dt} = u \pm a$$

(e) Now

$$dR = a\frac{d\rho}{\rho}$$

But for isentropic flow we have: $dp = a^2 d\rho$, therefore

$$dR = \frac{dp}{\rho a}$$

For an isentropic process $p = C\rho^{\gamma}$. Taking the logarithmic derivative of this expression gives

$$\frac{dp}{p} = \gamma\frac{d\rho}{\rho}$$

So

$$\frac{dp}{d\rho} = \gamma\frac{p}{\rho} = \gamma p\rho^{-1} = \gamma C^{\frac{1}{\gamma}} pp^{-\frac{1}{\gamma}} = \gamma C^{\frac{1}{\gamma}} p^{\frac{\gamma-1}{\gamma}} = a^2$$

Taking the logarithmic derivative of this expression brings

$$2\frac{da}{a} = \frac{\gamma - 1}{\gamma}\frac{dp}{p}$$

Thus,

$$dp = \frac{2\gamma}{\gamma - 1}\frac{p}{a}da = \frac{2}{\gamma - 1}\rho a da = \rho a dR$$

So

$$dR = \frac{2}{\gamma - 1}da$$

Integration using the given initial conditions produces

$$R = \frac{2}{\gamma - 1}a$$

Problem 6. – Obtain the characteristic equations of the pair of pde in part (b) of problem 5 by using the Method of Indeterminate Derivatives.

In addition to the given set of equations

$$\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial R}{\partial x} = 0$$

we have the total derivatives of R and u

$$dt\frac{\partial R}{\partial t} + dx\frac{\partial R}{\partial x} = dR$$
$$dt\frac{\partial u}{\partial t} + dx\frac{\partial u}{\partial x} = du$$

The above provide four equations in terms of the four unknown derivatives. Solving for any of the derivatives (here $\partial R/\partial t$ was selected) using Cramer's Rule yields

$$\frac{\partial R}{\partial t} = \frac{|N|}{|D|}$$

where

$$|N| = \begin{vmatrix} 0 & u & 0 & a \\ 0 & a & 1 & u \\ dR & dx & 0 & 0 \\ du & 0 & dt & dx \end{vmatrix}$$

$$|D| = \begin{vmatrix} 1 & u & 0 & a \\ 0 & a & 1 & u \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix}$$

Setting the determinant D to zero gives the equation of the characteristic in the $x - t$ plane, i.e.,

$$|D| = \begin{vmatrix} 1 & u & 0 & a \\ 0 & a & 1 & u \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix} = \begin{vmatrix} a & 1 & u \\ dx & 0 & 0 \\ 0 & dt & dx \end{vmatrix} - u\begin{vmatrix} 0 & 1 & u \\ dt & 0 & 0 \\ 0 & dt & dx \end{vmatrix} - a\begin{vmatrix} 0 & a & 1 \\ dt & dx & 0 \\ 0 & 0 & dt \end{vmatrix}$$

$$= -dx(dx - udt) + udt(dx - udt) - adt(0 - adt)$$

$$= -(dx)^2 + udxdt + udxdt - u^2(dt)^2 + a^2(dt)^2 = 0$$

which produces the following quadratic

$$\left(\frac{dx}{dt}\right)^2 - 2u\left(\frac{dx}{dt}\right) + u^2 - a^2 = 0$$

Solving this gives

$$\frac{dx}{dt} = u \pm \sqrt{u^2 - (u^2 - a^2)} = u \pm a$$

The compatibility equation is obtained by equating the determinant N to zero, i.e.,

$$|N| = \begin{vmatrix} 0 & u & 0 & a \\ 0 & a & 1 & u \\ dR & dx & 0 & 0 \\ du & 0 & dt & dx \end{vmatrix} = dR\begin{vmatrix} u & 0 & a \\ a & 1 & u \\ 0 & dt & dx \end{vmatrix} - du\begin{vmatrix} u & 0 & a \\ a & 1 & u \\ dx & 0 & 0 \end{vmatrix}$$

$$= dR[u(dx - udt) - a(0 - adt)] - dudx(0 - a)$$

$$= dR(udx - u^2dt + a^2dt) + adudx = 0$$

which produces

$$\left(u\frac{dx}{dt} + a^2 - u^2\right)dR + \left(a\frac{dx}{dt}\right)du = 0$$

Replace dx/dt with $u \pm a$ yields

$$[u(u \pm a) + a^2 - u^2]dR + [a(u \pm a)]du = (\pm ua + a^2)dR + (ua \pm a^2)du = 0$$

which reduces to

$$\pm dR + du = 0$$

along lines with slopes

$$\frac{dx}{dt} = u \pm a$$

Problem 7. – Obtain the characteristic equations of the pair of pde in part (b) of problem 5 by using the Method of Linear Combination.

Multiply the given equations

$$\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial R}{\partial x} = 0$$

by σ_1 and σ_2, respectively, and add to get

$$\sigma_1\left(\frac{\partial R}{\partial t}+u\frac{\partial R}{\partial x}+a\frac{\partial u}{\partial x}\right)+\sigma_2\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+a\frac{\partial R}{\partial x}\right)=0$$

Rearrangement brings

$$\sigma_1\left[\frac{\partial R}{\partial t}+\left(u+\frac{\sigma_2}{\sigma_1}a\right)\frac{\partial R}{\partial x}\right]+\sigma_2\left[\frac{\partial u}{\partial t}+\left(u+\frac{\sigma_1}{\sigma_2}a\right)\frac{\partial u}{\partial x}\right]=0$$

Compare the group of terms within the square brackets to the following total derivatives

$$\frac{\partial R}{\partial t}+\frac{dx}{dt}\frac{\partial R}{\partial x}=\frac{dR}{dt}$$

$$\frac{\partial u}{\partial t}+\frac{dx}{dt}\frac{\partial u}{\partial x}=\frac{du}{dt}$$

From this comparison we may write the slope, dx/dt, denoted as λ, as

$$\frac{dv}{du}=\lambda=u+\frac{\sigma_2}{\sigma_1}a=u+\frac{\sigma_1}{\sigma_2}a$$

Expanding this pair of equations produces two equations for σ_1 and σ_2

$$(u-\lambda)\sigma_1+a\sigma_2=0$$

$$a\sigma_1+(u-\lambda)\sigma_2=0$$

A unique solution for σ_1 and σ_2 will be obtained if and only if the determinant of the coefficients vanishes, i.e.,

$$\begin{vmatrix}(u-\lambda) & a\\ a & (u-\lambda)\end{vmatrix}=0$$

Expanding and rearranging the result produces the quadratic equation,

$$(u-\lambda)^2-a^2=0$$

Solution of this expression yields

$$\lambda=\frac{dx}{dt}=u\pm a$$

To derive the compatibility equation, incorporate the total derivative equations into the combined equation and obtain

$$\sigma_1 dR + \sigma_2 du = 0$$

or

$$\frac{\sigma_1}{\sigma_2} dR + du = 0$$

The relationship between σ_1 and σ_2 is obtained from use of either

$$(u - \lambda)\sigma_1 + a\sigma_2 = 0$$

$$a\sigma_1 + (u - \lambda)\sigma_2 = 0$$

Using either of these and the expression for λ produces

$$\frac{\sigma_1}{\sigma_2} = \frac{\lambda - u}{a} = \frac{u \pm a - u}{a} = \pm 1$$

Hence,

$$\pm dR + du = 0$$

along lines with slopes

$$\frac{dx}{dt} = u \pm a$$

Problem 8. – Obtain the characteristic equations of the pair of pde in part (b) of problem 5 by using Eigenanalysis.

The pair of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial R}{\partial x} = 0$$

$$\frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} + a\frac{\partial u}{\partial x} = 0$$

can be written in vector matrix form as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial t}+\begin{bmatrix} u & a \\ a & u \end{bmatrix}\frac{\partial \mathbf{w}}{\partial x}=0$$

where the dependent column vector is $\mathbf{w} = \begin{bmatrix} u \\ R \end{bmatrix}$. Defining the coefficient matrices as

$$\mathbf{A}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix} u & a \\ a & u \end{bmatrix}$$

the above equation may be written as

$$\mathbf{A}\frac{\partial \mathbf{w}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{w}}{\partial x}=0$$

Note **A** is the identity matrix **I**. Therefore, $\mathbf{A}^{-1}$ = I. So

$$\frac{\partial \mathbf{w}}{\partial t}+\mathbf{A}^{-1}\mathbf{B}\frac{\partial \mathbf{w}}{\partial x}=\frac{\partial \mathbf{w}}{\partial t}+\mathbf{B}\frac{\partial \mathbf{w}}{\partial x}=0$$

The characteristic directions are obtained by determining the *eigenvalues* of the coefficient matrix of $\partial \mathbf{w}/\partial x$, which in this case is matrix **B**, hence

$$\left|\mathbf{B}-\lambda\mathbf{I}\right|=0$$

that is

$$\begin{vmatrix} u-\lambda & a \\ a & u-\lambda \end{vmatrix}=0$$

This is the same determinant as obtained in the previous problem. Expanding gives

$$(u-\lambda)^2-a^2=0$$

Solution of this expression yields

$$\lambda=\frac{dx}{dt}=u\pm a$$

To derive the compatibility equation, the left eigenvectors must first be determined. Rather than obtaining results for each characteristic. The following applies to characteristics of either family

$$\mathbf{l}^{T}(\mathbf{B}-\lambda\mathbf{I})=0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} u-\lambda & a \\ a & u-\lambda \end{bmatrix} = 0$$

Expanding gives two equations produces

$$(u-\lambda)l_1 + al_2 = 0$$

$$al_1 + (u-\lambda)l_2 = 0$$

Hence,

$$l_2 = \left(\frac{\lambda - u}{a}\right)l_1$$

$$l_2 = \left(\frac{a}{\lambda - u}\right)l_1$$

Using the fact that

$$\lambda = u \pm a$$

produces that

$$l_2 = \pm l_1$$

Take l_1 to be unity. The compatibility equation for characteristics is

$$\mathbf{l}^T d\mathbf{w} = 0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} du \\ dR \end{bmatrix} = \begin{bmatrix} 1 & \pm 1 \end{bmatrix}\begin{bmatrix} du \\ dR \end{bmatrix} = du \pm dR = 0$$

Problem 9. – (a) Combine Eq.(14.29) with both expressions in Eq.(14.27) to obtain Eq.(14.30); (b) Substitute Eq.(14.30) into Eq.(14.29) to obtain

$$\frac{dv}{du} = \frac{-C_2 \mp \sqrt{C_2^2 - C_1 C_3}}{C_3}$$

(a) We begin with the simpler of the two expressions

$$(C_1\lambda - C_2)\sigma_1 + \sigma_2 = 0$$

Hence,

$$\frac{\sigma_2}{\sigma_1} = C_2 - C_1\lambda$$

But

$$\lambda = \frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1}$$

Substitution brings

$$\frac{\sigma_2}{\sigma_1} = C_2 - C_1\left(\frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_1}\right) = \mp\sqrt{C_2^2 - C_1C_3}$$

Next the second expression is used, i.e.,

$$(C_2\lambda - C_3)\sigma_1 + \lambda\sigma_2 = 0$$

or

$$\frac{\sigma_2}{\sigma_1} = \frac{C_3 - C_2\lambda}{\lambda} = \frac{C_3}{\lambda} - C_2$$

$$= \frac{C_1C_3}{C_2 \pm \sqrt{C_2^2 - C_1C_3}} - C_2 = \frac{C_1C_3 - C_2\left(C_2 \pm \sqrt{C_2^2 - C_1C_3}\right)}{C_2 \pm \sqrt{C_2^2 - C_1C_3}}$$

$$= \frac{-\left(\sqrt{C_2^2 - C_1C_3}\right)^2 \mp C_2\sqrt{C_2^2 - C_1C_3}}{C_2 \pm \sqrt{C_2^2 - C_1C_3}} = \mp C_2\sqrt{C_2^2 - C_1C_3}\,\frac{C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_2 \pm \sqrt{C_2^2 - C_1C_3}}$$

$$= \mp C_2\sqrt{C_2^2 - C_1C_3}$$

(b) From Eq.(14.29) we have

$$C_1du + \left(C_2 + \frac{\sigma_2}{\sigma_1}\right)dv = 0$$

Substituting the results from part (a) gives

$$C_1du + \left(C_2 \mp \sqrt{C_2^2 - C_1C_3}\right)dv = 0$$

Rearranging this brings

$$\frac{dv}{du} = \frac{-C_1}{C_2 \pm \sqrt{C_2^2 - C_1C_3}} = \frac{-C_1}{C_2 \pm \sqrt{C_2^2 - C_1C_3}} \frac{C_2 \mp \sqrt{C_2^2 - C_1C_3}}{C_2 \mp \sqrt{C_2^2 - C_1C_3}}$$

$$= \frac{-C_1\left(C_2 \mp \sqrt{C_2^2 - C_1C_3}\right)}{C_2^2 - \left(C_2^2 - C_1C_3\right)} = \frac{-C_2 \pm \sqrt{C_2^2 - C_1C_3}}{C_3}$$

Problem 10. – In example 14.2 only one of four compatibility equations was determined. Complete this example by determine the remaining three.

The complete left eigenvector is

$$\mathbf{L} = \begin{bmatrix} \rho u & 0 & \dfrac{-1}{\sqrt{\left(u^2+v^2\right)-a^2}} & \dfrac{1}{\sqrt{\left(u^2+v^2\right)-a^2}} \\ \rho v & 0 & \dfrac{u/v}{\sqrt{\left(u^2+v^2\right)-a^2}} & \dfrac{-u/v}{\sqrt{\left(u^2+v^2\right)-a^2}} \\ 1 & -1/a^2 & 1/\rho va & 1/\rho va \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The compatibility equations are established by application of

$$\mathbf{l}_{\lambda \mathbf{i}}^{\mathbf{T}} d\mathbf{w} = 0$$

for each left eigenvector corresponding to a particular eigenvalue.

Along the characteristic given by dy/dy = λ_2

$$\mathbf{l}_{\lambda 2}^{\mathbf{T}} d\mathbf{w} = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & l_4^2 \end{bmatrix} \begin{bmatrix} du \\ dv \\ dp \\ d\rho \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{-1}{a^2} & 1 \end{bmatrix} \begin{bmatrix} du \\ dv \\ dp \\ d\rho \end{bmatrix} = -\frac{dp}{a^2} + d\rho = 0$$

So we obtain the speed sound expression

$$\frac{dp}{d\rho} = a^2$$

Along the characteristics given by dy/dy = λ_3 and λ_4 we have

$$\mathbf{l}_{\lambda3,4}^{\mathbf{T}}\mathbf{dw} = \begin{bmatrix} l_1^{3,4} & l_2^{3,4} & l_3^{3,4} & l_4^{3,4} \end{bmatrix}\begin{bmatrix} du \\ dv \\ dp \\ d\rho \end{bmatrix} = \begin{bmatrix} \dfrac{\mp 1}{a\beta} & \dfrac{\pm u/v}{a\beta} & \dfrac{1}{\rho va} & 0 \end{bmatrix}\begin{bmatrix} du \\ dv \\ dp \\ d\rho \end{bmatrix} = 0$$

where $\beta = \sqrt{M^2 - 1} = \sqrt{\left(u^2 + v^2\right)/a^2 - 1}$. Expanding and canceling terms yields

$$\mp vdu \pm udv + \frac{\beta}{\rho}dp = 0$$

Now according to Bernoulli's equation

$$\frac{dp}{\rho} = -(udu + vdv)$$

Uniting the expressions and rearranging produces

$$\frac{dv}{du} = \frac{v \pm \beta u}{u \mp \beta v}$$

Multiply both numerator and denominator by u ± βv. The numerator simplifies as follows

$$(v \pm \beta u)(u \pm \beta v) = uv \pm \beta u^2 \pm \beta v^2 + uv\beta^2$$

$$= uv\left(1 + \beta^2\right) \pm \beta\left(u^2 + v^2\right) = \frac{\left(u^2 + v^2\right)}{a^2}\left[uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}\right]$$

Whereas the denominator simplifies as follows

$$(u \mp \beta u)(u \pm \beta v) = u^2 - \beta^2 v^2 = u^2 - \frac{\left(u^2 + v^2\right)}{a^2}v^2 + v^2 = \frac{\left(u^2 + v^2\right)}{a^2}\left(a^2 - v^2\right)$$

Hence,

$$\frac{dv}{du} = \frac{uv \pm \sqrt{a^2\left(u^2 + v^2\right) - a^4}}{a^2 - v^2}$$

Problem 11. – Use the Method of Indeterminate Derivatives to determine the equations of the characteristics for linearized, two-dimensional, supersonic flow described by

$$\left(1 - M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

There is no need to go through the entire analysis since this is a simple extension of theory presented in the Chapter. Instead simply let

$$C_1 = \left(1 - M_\infty^2\right) = -\beta^2$$

$$C_2 = 0$$

$$C_3 = 1$$

Note for supersonic flow C_1 is negative and the potential equation is actually the wave equation, which is hyperbolic.

$$\frac{dy}{dx} = \frac{C_2 \pm \sqrt{C_2^2 - C_1 C_3}}{C_1} = \frac{0 \pm \sqrt{0 + \beta^2}}{-\beta^2} = \frac{\mp 1}{\beta} = \frac{\mp 1}{\sqrt{M_\infty^2 - 1}}$$

$$\frac{dv}{du} = \frac{-C_2 \mp \sqrt{C_2^2 - C_1 C_3}}{C_3} = \frac{0_2 \mp \sqrt{0 + \beta^2}}{1} = \mp\beta = \mp\sqrt{M_\infty^2 - 1}$$

Problem 12. – (a) Show that each dependent variable in the following pair of equations must satisfy the wave equation and therefore the set is hyperbolic

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

(b) Use the Method of Linear Combination to determine the equations of the characteristics for this set of equations.

(a) The wave equation is written in the x-y plane as

$$\frac{\partial^2 f}{\partial x^2} = c^2 \frac{\partial^2 f}{\partial y^2}$$

where c is the wave speed.

Now differentiate the first equation wrt to x and the second wrt y to obtain

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

Subtraction produces the wave equation with a wave speed of ± 1

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

Now differentiate the first equation wrt to y and the second wrt x to obtain

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} = 0$$

Subtraction produces the wave equation with a wave speed of ± 1

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

(b) Multiply the given equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

by σ_1 and σ_2, respectively, and add to get

$$\sigma_1\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \sigma_2\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

Rearrangement brings

$$\sigma_1\left[\frac{\partial u}{\partial x}+\left(\frac{\sigma_2}{\sigma_1}\right)\frac{\partial u}{\partial y}\right]-\sigma_2\left[\frac{\partial v}{\partial x}+\left(\frac{\sigma_1}{\sigma_2}\right)\frac{\partial v}{\partial y}\right]=0$$

Compare the group of terms within the square brackets to the following total derivatives

$$\frac{\partial u}{\partial x}+\frac{dy}{dx}\frac{\partial u}{\partial y}=\frac{du}{dx}$$

$$\frac{\partial v}{\partial x}+\frac{dy}{dx}\frac{\partial v}{\partial y}=\frac{dv}{dx}$$

From this comparison we may write the slope, dx/dt, denoted as λ, as

$$\frac{dv}{du}=\lambda=\frac{\sigma_2}{\sigma_1}=\frac{\sigma_1}{\sigma_2}$$

Expanding this pair of equations produces two equations for σ_1 and σ_2

$$\lambda\sigma_1-\sigma_2=0$$

$$\sigma_1-\lambda\sigma_2=0$$

A unique solution for σ_1 and σ_2 will be obtained if and only if the determinant of the coefficients vanishes, i.e.,

$$\begin{vmatrix}\lambda & -1\\ 1 & -\lambda\end{vmatrix}=0$$

Expanding and rearranging the result produces the quadratic equation,

$$-\lambda^2+1=0$$

Solution of this expression yields

$$\lambda=\frac{dx}{dt}=\pm 1$$

The eigenvalues yield the wave speed of the wave equation.

To derive the compatibility equation, incorporate the total derivative equations into the combined equation and obtain

$$\sigma_1 du-\sigma_2 dv=0$$

or

$$\frac{\sigma_1}{\sigma_2}du - dv = 0$$

The relationship between σ_1 and σ_2 is obtained from use of either

$$(u-\lambda)\sigma_1 + a\sigma_2 = 0$$

$$a\sigma_1 + (u-\lambda)\sigma_2 = 0$$

Using either of these and the expression for λ produces

$$\frac{\sigma_1}{\sigma_2} = \frac{\lambda - u}{a} = \frac{u \pm a - u}{a} = \pm 1$$

Hence,

$$\pm\, du - dv = 0$$

or

$$\frac{dv}{du} = \pm 1$$

along lines with slopes

$$\frac{dx}{dt} = u \pm a$$

Problem 13. – Use Eigenanalysis to determine the equations of the characteristics in problem 12.

The pair of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

can be written in vector matrix form as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial t} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial x} = 0$$

where the dependent column vector is $\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}$. Defining the coefficient matrices as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

the above equation may be written as

$$\mathbf{A}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{w}}{\partial x} = 0$$

The inverse of **A** is

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}^{-1}\mathbf{B}\frac{\partial \mathbf{w}}{\partial x} = \frac{\partial \mathbf{w}}{\partial t} + \mathbf{C}\frac{\partial \mathbf{w}}{\partial x} = 0$$

where

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The characteristic directions are obtained by determining the *eigenvalues* of **C**

$$\left|\mathbf{C} - \lambda\mathbf{I}\right| = 0$$

Therefore,

$$\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0$$

Expanding gives

$$\lambda^2 - 1^2 = 0$$

Solution yields

$$\lambda = \frac{dy}{dx} = \pm 1$$

To derive the compatibility equation, the left eigenvectors must first be determined. Rather than obtaining results for each characteristic. The following applies to characteristics of either family

$$\mathbf{l}^{\mathrm{T}}\left(\mathbf{C} - \lambda\mathbf{I}\right) = 0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = 0$$

Expanding gives two equations produces

$$\lambda l_1 + l_2 = 0$$

$$l_1 + \lambda l_2 = 0$$

Hence,

$$l_2 = -\lambda l_1 = \mp l_1$$

$$l_2 = -\frac{1}{\lambda} l_1 = \mp l_1$$

where the slope of the characteristics ($\lambda = \pm 1$) has been used. So

$$l_2 = \pm l_1$$

Take l_1 to be unity. The compatibility equation for characteristics is

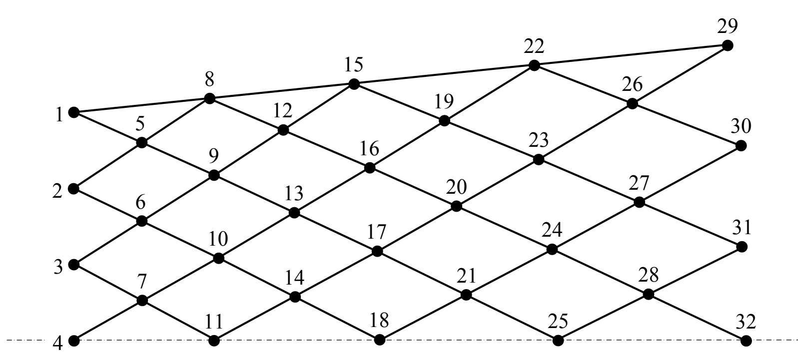
$$\mathbf{l}^{\mathbf{T}} d\mathbf{w} = 0$$

or

$$\begin{bmatrix} l_1 & l_2 \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 1 & \pm 1 \end{bmatrix}\begin{bmatrix} du \\ dv \end{bmatrix} = du \pm dv = 0$$

Problem 14. – Complete the solution of Example 14.3 by determining the solution at points 19 to 32. Check the accuracy of the results.

The numbering of the points is contained in the following figure



A set of tables that contains data for all of the labeled points follows:

point	$\alpha =$ $(C_I+C_{II})/2$ deg	$\nu =$ $(C_I - C_{II})/2$ deg	$C_I =$ $\alpha + \nu$ deg	$C_{II} =$ $\alpha - \nu$ deg	M	μ deg	$\alpha + \mu$ deg	$\alpha - \mu$ deg
1	6.00	26.3798	32.3798	20.3798	2.0000	30.0000	36.0000	-24.0000
2	4.00	26.3798	30.3798	22.3798	2.0000	30.0000	34.0000	-26.0000
3	2.00	26.3798	28.3798	24.3798	2.0000	30.0000	32.0000	-28.0000
4	0.00	26.3798	26.3798	26.3798	2.0000	30.0000	30.0000	-30.0000
5	5.00	27.3798	32.3798	22.3798	2.0365	29.4095	34.4095	-24.4095
6	3.00	27.3798	30.3798	24.3798	2.0365	29.4095	32.4095	-26.4095
7	1.00	27.3798	28.3798	26.3798	2.0365	29.4095	30.4095	-28.4095
8	6.00	28.3798	34.3798	22.3798	2.0733	28.8370	34.8370	-22.8370
9	4.00	28.3798	32.3798	24.3798	2.0733	28.8370	32.8370	-24.8370
10	2.00	28.3798	30.3798	26.3798	2.0733	28.8370	30.8370	-26.8370
11	0.00	28.3798	28.3798	28.3798	2.0733	28.8370	28.8370	-28.8370
12	5.00	29.3798	34.3798	24.3798	2.1106	28.2815	33.2815	-23.2815
13	3.00	29.3798	32.3798	26.3798	2.1106	28.2815	31.2815	-25.2815
14	1.00	29.3798	30.3798	28.3798	2.1106	28.2815	29.2815	-27.2815
15	6.00	30.3798	36.3798	24.3798	2.1483	27.7419	33.7419	-21.7419
16	4.00	30.3798	34.3798	26.3798	2.1483	27.7419	31.7419	-23.7419
17	2.00	30.3798	32.3798	28.3798	2.1483	27.7419	29.7419	-25.7419
18	0.00	30.3798	30.3798	30.3798	2.1483	27.7419	27.7419	-27.7419
19	5.00	31.3798	36.3798	26.3798	2.1864	27.2173	32.2173	-22.2173
20	3.00	31.3798	34.3798	28.3798	2.1864	27.2173	30.2173	-24.2173
21	1.00	31.3798	32.3798	30.3798	2.1864	27.2173	28.2173	-26.2173
22	6.00	32.3798	38.3798	26.3798	2.2251	26.7068	32.7068	-20.7068
23	4.00	32.3798	36.3798	28.3798	2.2251	26.7068	30.7068	-22.7068
24	2.00	32.3798	34.3798	30.3798	2.2251	26.7068	28.7068	-24.7068
25	0.00	32.3798	32.3798	32.3798	2.2251	26.7068	26.7068	-26.7068
26	5.00	33.3798	38.3798	28.3798	2.2642	26.2096	31.2096	-21.2096
27	3.00	33.3798	36.3798	30.3798	2.2642	26.2096	29.2096	-23.2096
28	1.00	33.3798	34.3798	32.3798	2.2642	26.2096	27.2096	-25.2096
29	6.00	34.3798	40.3798	28.3798	2.3039	25.7250	31.7250	-19.7250
30	4.00	34.3798	38.3798	30.3798	2.3039	25.7250	29.7250	-21.7250
31	2.00	34.3798	36.3798	32.3798	2.3039	25.7250	27.7250	-23.7250
32	0.00	34.3798	34.3798	34.3798	2.3039	25.7250	25.7250	-25.7250

The coordinates and slopes of the characteristics in the physical plane are contained in the following table

point	α deg	μ	$\alpha + \mu$	$\alpha - \mu$	m_I	m_{II}	x	y
1	6	30	36	-24			9.5144	1.0000
2	4	30	34	-26			9.5435	0.6673
3	2	30	32	-28			9.5609	0.3339
4	0	30	30	-30			9.5668	0.0000
5	5	29.4095	34.4095	-24.4095	-0.4495	0.6797	9.8265	0.8597
6	3	29.4095	32.4095	-26.4095	-0.4922	0.6298	9.8505	0.5162
7	1	29.4095	30.4095	-28.4095	-0.5363	0.5821	9.8625	0.1722
8	6	28.8370	34.8370	-22.8370	0.1051	0.6905	10.1222	1.0639
9	4	28.8370	32.8370	-24.8370	-0.4750	0.6106	10.1563	0.7030
10	2	28.8370	30.8370	-26.8370	-0.5186	0.6401	10.1541	0.3588
11	0	28.8370	28.8370	-28.8370	-0.5457	0.0000	10.1779	0.0000
12	5	28.2815	33.2815	-23.2815	-0.4257	0.6509	10.4780	0.9124
13	3	28.2815	31.2815	-25.2815	-0.4676	0.6023	10.4768	0.5532
14	1	28.2815	29.2815	-27.2815	-0.5108	0.5557	10.5030	0.1806
15	6	27.7419	33.7419	-21.7419	0.1051	0.6622	10.8171	1.1369
16	4	27.7419	31.7419	-23.7419	-0.4351	0.6131	10.8201	0.7636
17	2	27.7419	29.7419	-25.7419	-0.4772	0.5660	10.8481	0.3760
18	0	27.7419	27.7419	-27.7419	-0.5208	0.0000	10.8497	0.0000
19	5.00	27.2173	32.2173	-22.2173	-0.4036	0.6244	11.1821	0.9896
20	3.00	27.2173	30.2173	-24.2173	-0.4448	0.5769	11.2153	0.5878
21	1.00	27.2173	28.2173	-26.2173	-0.4873	0.5313	11.2181	0.1957
22	6.00	26.7068	32.7068	-20.7068	0.1051	0.6361	11.5317	1.2120
23	4.00	26.7068	30.7068	-22.7068	-0.4134	0.5882	11.6028	0.8157
24	2.00	26.7068	28.7068	-24.7068	-0.4549	0.5421	11.6101	0.4082
25	0.00	26.7068	26.7068	-26.7068	-0.4978	0.0000	11.6112	0.0000
26	5.00	26.2096	31.2096	-21.2096	-0.3830	0.5999	11.9783	1.0410
27	3.00	26.2096	29.2096	-23.2096	-0.4236	0.5534	12.0240	0.6373
28	1.00	26.2096	27.2096	-25.2096	-0.4654	0.5086	12.0298	0.2129
29	6.00	25.7250	31.7250	-19.7250	0.1051	0.6120	12.4084	1.3042
30	4.00	25.7250	29.7250	-21.7250	-0.3933	0.5650	12.4265	0.8647
31	2.00	25.7250	27.7250	-23.7250	-0.4341	0.5198	12.4720	0.4428
32	0.00	25.7250	25.7250	-25.7250	-0.4763	0.0000	12.4767	0.0000

The accuracy of the calculations is assessed in the following table

Point	Area	A/A*	M (exact solution)	M (MOC)	% Error
4	A_4	1.6875	2	2	
11	$1.06395A_4$	1.7953	2.0733	2.0733	0
18	$1.1341A_4$	1.9138	2.1472	2.1483	0.0514
25	$1.2137A_4$	2.0471	2.2239	2.2251	0.0506
32	$1.3042A_4$	2.2008	2.3038	2.3039	0.0011

Problem 15. – Using the same number of points repeat Example 14.3 for:
(a) $M_{initial} = 2.0$, total wedge angle of 12˚ and $\gamma = 1.3$;
(b) $M_{initial} = 4.0$, total wedge angle of 12˚ and $\gamma = 1.4$;
(c) $M_{initial} = 2.0$, total wedge angle of 24˚ and $\gamma = 1.4$

(a) $M_{initial} = 2.0$, total wedge angle of 12˚ and $\gamma = 1.3$

point	α deg	ν	$C(+)_I$	$C(-)_{II}$	M	μ	α + μ	α − μ
1	6.00	28.6809	34.6809	22.6809	2.0000	30.0000	36.0000	-24.0000
2	4.00	28.6809	32.6809	24.6809	2.0000	30.0000	34.0000	-26.0000
3	2.00	28.6809	30.6809	26.6809	2.0000	30.0000	32.0000	-28.0000
4	0.00	28.6809	28.6809	28.6809	2.0000	30.0000	30.0000	-30.0000
5	5.00	29.6809	34.6809	24.6809	2.0324	29.4747	34.4747	-24.4747
6	3.00	29.6809	32.6809	26.6809	2.0324	29.4747	32.4747	-26.4747
7	1.00	29.6809	30.6809	28.6809	2.0324	29.4747	30.4747	-28.4747
8	6.00	30.6809	36.6809	24.6809	2.0649	28.9650	34.9650	-22.9650
9	4.00	30.6809	34.6809	26.6809	2.0649	28.9650	32.9650	-24.9650
10	2.00	30.6809	32.6809	28.6809	2.0649	28.9650	30.9650	-26.9650
11	0.00	30.6809	30.6809	30.6809	2.0649	28.9650	28.9650	-28.9650
12	5.00	31.6809	36.6809	26.6809	2.0978	28.4698	33.4698	-23.4698
13	3.00	31.6809	34.6809	28.6809	2.0978	28.4698	31.4698	-25.4698
14	1.00	31.6809	32.6809	30.6809	2.0978	28.4698	29.4698	-27.4698
15	6.00	32.6809	38.6809	26.6809	2.1309	27.9884	33.9884	-21.9884
16	4.00	32.6809	36.6809	28.6809	2.1309	27.9884	31.9884	-23.9884
17	2.00	32.6809	34.6809	30.6809	2.1309	27.9884	29.9884	-25.9884
18	0.00	32.6809	32.6809	32.6809	2.1309	27.9884	27.9884	-27.9884

point	α deg	μ	α + μ	α − μ	m_I	m_{II}	x	y
1	6	30	36	-24			9.5144	1.0000
2	4	30	34	-26			9.5435	0.6673
3	2	30	32	-28			9.5609	0.3339
4	0	30	30	-30			9.5668	0.0000
5	5	29.4747	34.4747	-24.4747	-0.4502	0.6806	9.8261	0.8597
6	3	29.4747	32.4747	-26.4747	-0.4929	0.6306	9.8501	0.5162
7	1	29.4747	30.4747	-28.4747	-0.5370	0.5829	9.8621	0.1721
8	6	28.9650	34.9650	-22.9650	0.1051	0.6929	10.1205	1.0637
9	4	28.9650	32.9650	-24.9650	-0.4764	0.6122	10.1551	0.7029
10	2	28.9650	30.9650	-26.9650	-0.5200	0.6425	10.1527	0.3589
11	0	28.9650	28.9650	-28.9650	-0.5479	0.0000	10.1763	0.0000
12	5	28.4698	33.4698	-23.4698	-0.4290	0.6548	10.4743	0.9120
13	3	28.4698	31.4698	-25.4698	-0.4709	0.6060	10.4732	0.5531
14	1	28.4698	29.4698	-27.4698	-0.5143	0.5593	10.4992	0.1806
15	6	27.9884	33.9884	-21.9884	0.1051	0.6677	10.8101	1.1362
16	4	27.9884	31.9884	-23.9884	-0.4396	0.6183	10.8129	0.7631
17	2	27.9884	29.9884	-25.9884	-0.4819	0.5711	10.8411	0.3758
18	0	27.9884	27.9884	-27.9884	-0.5257	0.0000	10.8429	0.0000

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Point	Area	A/A*	M (Exact Solution)	M (MOC)	% Error
4	A_4	1.7732	2	2	
11	$1.0637A_4$	1.8862	2.0649	2.0649	0
18	$1.1334A_4$	2.0097	2.1299	2.1309	0.0438

(b) $M_{initial} = 4.0$, total wedge angle of 12° and $\gamma = 1.4$

point	α deg	ν	$C(+)_I$	$C(-)_{II}$	M	μ	$\alpha + \mu$	$\alpha - \mu$
1	6.00	65.7848	71.7848	59.7848	4.0000	14.4775	20.4775	-8.4775
2	4.00	65.7848	69.7848	61.7848	4.0000	14.4775	18.4775	-10.4775
3	2.00	65.7848	67.7848	63.7848	4.0000	14.4775	16.4775	-12.4775
4	0.00	65.7848	65.7848	65.7848	4.0000	14.4775	14.4775	-14.4775
5	5.00	66.7848	71.7848	61.7848	4.0768	14.1991	19.1991	-9.1991
6	3.00	66.7848	69.7848	63.7848	4.0768	14.1991	17.1991	-11.1991
7	1.00	66.7848	67.7848	65.7848	4.0768	14.1991	15.1991	-13.1991
8	6.00	67.7848	73.7848	61.7848	4.1557	13.9238	19.9238	-7.9238
9	4.00	67.7848	71.7848	63.7848	4.1557	13.9238	17.9238	-9.9238
10	2.00	67.7848	69.7848	65.7848	4.1557	13.9238	15.9238	-11.9238
11	0.00	67.7848	67.7848	67.7848	4.1557	13.9238	13.9238	-13.9238
12	5.00	68.7848	73.7848	63.7848	4.2370	13.6516	18.6516	-8.6516
13	3.00	68.7848	71.7848	65.7848	4.2370	13.6516	16.6516	-10.6516
14	1.00	68.7848	69.7848	67.7848	4.2370	13.6516	14.6516	-12.6516
15	6.00	69.7848	75.7848	63.7848	4.3207	13.3822	19.3822	-7.3822
16	4.00	69.7848	73.7848	65.7848	4.3207	13.3822	17.3822	-9.3822
17	2.00	69.7848	71.7848	67.7848	4.3207	13.3822	15.3822	-11.3822
18	0.00	69.7848	69.7848	69.7848	4.3207	13.3822	13.3822	-13.3822

point	α deg	μ	$\alpha + \mu$	$\alpha - \mu$	m_I	m_{II}	x	y
1	6	14	20	-8			9.5144	1.0000
2	4	14	18	-10			9.5435	0.6673
3	2	14	16	-12			9.5609	0.3339
4	0	14	14	-14			9.5668	0.0000
5	5	14.1991	19.1991	-9.1991	-0.1555	0.3412	10.2041	0.8927
6	3	14.1991	17.1991	-11.1991	-0.1915	0.3026	10.2291	0.5361
7	1	14.1991	15.1991	-13.1991	-0.2279	0.2649	10.2416	0.1788
8	6	13.9238	19.9238	-7.9238	0.1051	0.3553	10.9225	1.1480
9	4	13.9238	17.9238	-9.9238	-0.1799	0.2905	10.9777	0.7536
10	2	13.9238	15.9238	-11.9238	-0.2162	0.3165	10.9073	0.3895
11	0	13.9238	13.9238	-13.9238	-0.2412	0.0000	10.9827	0.0000
12	5	13.6516	18.6516	-8.6516	-0.1457	0.3305	11.7892	1.0218
13	3	13.6516	16.6516	-10.6516	-0.1815	0.2922	11.7029	0.6219
14	1	13.6516	14.6516	-12.6516	-0.2178	0.2547	11.7722	0.2011
15	6	13.3822	19.3822	-7.3822	0.1051	0.3447	12.6965	1.3345
16	4	13.3822	17.3822	-9.3822	-0.1587	0.3061	12.5927	0.8942
17	2	13.3822	15.3822	-11.3822	-0.1947	0.2683	12.6522	0.4371
18	0	13.3822	13.3822	-13.3822	-0.2312	0.0000	12.6420	0.0000

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Point	Area	A/A*	M (Exact Solution)	M (MOC)	% Error
4	A_4	10.7188	4.00	4.0000	
11	$1.0637A_4$	12.3051	4.1558	4.1557	-0.0014
18	$1.1334A_4$	14.1642	4.3172	4.3207	0.0799

(c) $M_{initial}$ = 2.0, total wedge angle of 24° and γ = 1.4

point	α deg	ν	$C(+)_I$	$C(-)_{II}$	M	μ	$\alpha+\mu$	$\alpha-\mu$
1	12.00	26.3798	38.3798	14.3798	2.0000	30.0000	42.0000	-18.0000
2	8.00	26.3798	34.3798	18.3798	2.0000	30.0000	38.0000	-22.0000
3	4.00	26.3798	30.3798	22.3798	2.0000	30.0000	34.0000	-26.0000
4	0.00	26.3798	26.3798	26.3798	2.0000	30.0000	30.0000	-30.0000
5	10.00	28.3798	38.3798	18.3798	2.0733	28.8370	38.8370	-18.8370
6	6.00	28.3798	34.3798	22.3798	2.0733	28.8370	34.8370	-22.8370
7	2.00	28.3798	30.3798	26.3798	2.0733	28.8370	30.8370	-26.8370
8	12.00	30.3798	42.3798	18.3798	2.1483	27.7419	39.7419	-15.7419
9	8.00	30.3798	38.3798	22.3798	2.1483	27.7419	35.7419	-19.7419
10	4.00	30.3798	34.3798	26.3798	2.1483	27.7419	31.7419	-23.7419
11	0.00	30.3798	30.3798	30.3798	2.1483	27.7419	27.7419	-27.7419
12	10.00	32.3798	42.3798	22.3798	2.2251	26.7068	36.7068	-16.7068
13	6.00	32.3798	38.3798	26.3798	2.2251	26.7068	32.7068	-20.7068
14	2.00	32.3798	34.3798	30.3798	2.2251	26.7068	28.7068	-24.7068
15	12.00	34.3798	46.3798	22.3798	2.3039	25.7250	37.7250	-13.7250
16	8.00	34.3798	42.3798	26.3798	2.3039	25.7250	33.7250	-17.7250
17	4.00	34.3798	38.3798	30.3798	2.3039	25.7250	29.7250	-21.7250
18	0.00	34.3798	34.3798	34.3798	2.3039	25.7250	25.7250	-25.7250

point	α deg	μ	$\alpha+\mu$	$\alpha-\mu$	m_I	m_{II}	x	y
1	12	30	42	-18			4.7046	1.0000
2	8	30	38	-22			4.7629	0.6694
3	4	30	34	-26			4.7980	0.3355
4	0	30	30	-30			4.8097	0.0000
5	10	28.8370	38.8370	-18.8370	-0.3330	0.7931	5.0393	0.8886
6	6	28.8370	34.8370	-22.8370	-0.4125	0.6852	5.0890	0.5349
7	2	28.8370	30.8370	-26.8370	-0.4968	0.5871	5.1139	0.1786
8	12	27.7419	39.7419	-15.7419	0.2126	0.8182	5.3407	1.1352
9	8	27.7419	35.7419	-19.7419	-0.3806	0.6454	5.4153	0.7455
10	4	27.7419	31.7419	-23.7419	-0.4628	0.7078	5.4084	0.3870
11	0	27.7419	27.7419	-27.7419	-0.5159	0.0000	5.4600	0.0000
12	10	26.7068	36.7068	-16.7068	-0.2910	0.7325	5.7749	1.0089
13	6	26.7068	32.7068	-20.7068	-0.3684	0.6303	5.7698	0.6148
14	2	26.7068	28.7068	-24.7068	-0.4499	0.5367	5.8288	0.1979
15	12	25.7250	37.7250	-13.7250	0.2126	0.7595	6.1746	1.3124
16	8	25.7250	33.7250	-17.7250	-0.3099	0.6548	6.1799	0.8834
17	4	25.7250	29.7250	-21.7250	-0.3882	0.5592	6.2447	0.4305
18	0	25.7250	25.7250	-25.7250	-0.4709	0.0000	6.2490	0.0000

.

Point	Area	A/A*	M (Exact Solution)	M (MOC)	% Error
4	A_4	1.6875	2.00	2.0000	
11	$1.0637A_4$	1.9157	2.1483	2.1483	-0.0004
18	$1.1334A_4$	2.1925	2.2997	2.3039	0.1827

Problem 16. – A supersonic flow at Mach 1.8 and $\gamma = 1.4$ enters the channel shown in Figure P14.16(a). Using the point-to-point method of characteristics, determine the Mach number distribution throughout the flow for the pattern shown in Figure P14.16(b).

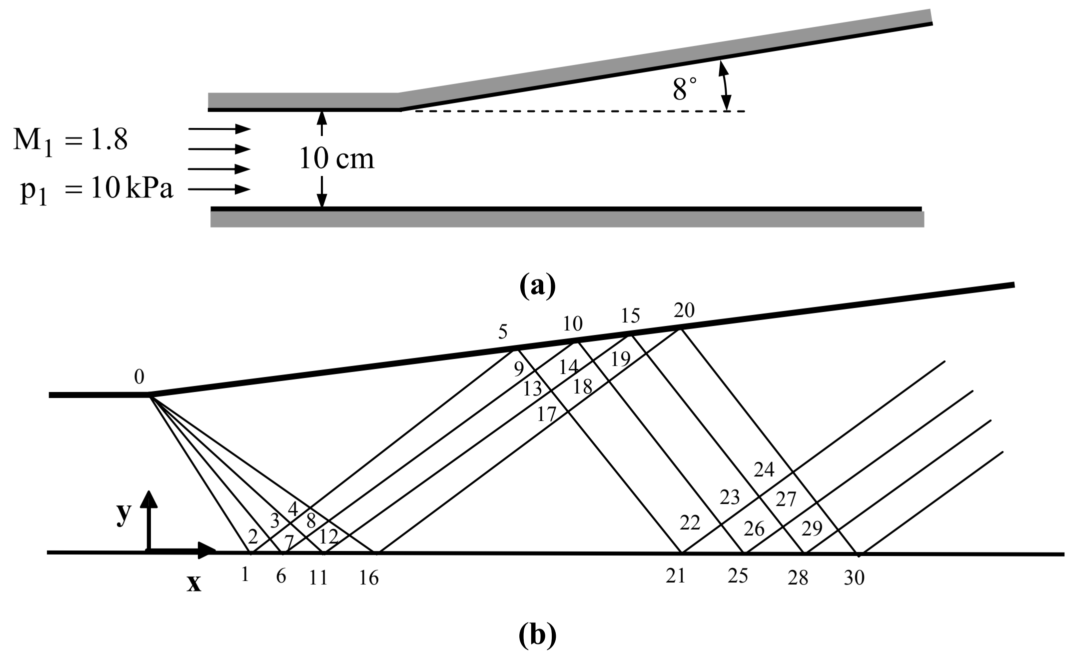


Figure P14.16

A spreadsheet program was constructed to solve this problem. Results of the program are contained within the following:

Input and computed initial data-

γ	γ-1/γ+1	α_1	M_1	wall ang	turns	Δ(angle)	x_o	y_o	p_1
1.4	0.1667	0	1.8	8	4	2	0	0.1	10

Results of calculations-

Method: Point-to-Point									
Note:	a shaded cell contains a value that is set								
Point	α	ν	$C_I = \nu+\alpha$	$C_{II} = \nu-\alpha$	M	μ	$\alpha + \mu$	$\alpha - \mu$	p/p_o
1	0.0	20.7251	20.7251	20.7251	1.8000	33.7490	33.7490	-33.7490	0.1740
2	2.0	22.7251	24.7251	20.7251	1.8697	32.3339	34.3339	-30.3339	0.1564
3	4.0	24.7251	28.7251	20.7251	1.9405	31.0204	35.0204	-27.0204	0.1402
4	6.0	26.7251	32.7251	20.7251	2.0125	29.7940	35.7940	-23.7940	0.1253
5	8.0	28.7251	36.7251	20.7251	2.0861	28.6433	36.6433	-20.6433	0.1117
6	0.0	24.7251	24.7251	24.7251	1.9405	31.0204	31.0204	-31.0204	0.1402
7	2.0	26.7251	28.7251	24.7251	2.0125	29.7940	31.7940	-27.7940	0.1253
8	4.0	28.7251	32.7251	24.7251	2.0861	28.6433	32.6433	-24.6433	0.1117
9	6.0	30.7251	36.7251	24.7251	2.1614	27.5591	33.5591	-21.5591	0.0993
10	8.0	32.7251	40.7251	24.7251	2.2385	26.5337	34.5337	-18.5337	0.0880
11	0.0	28.7251	28.7251	28.7251	2.0861	28.6433	28.6433	-28.6433	0.1117
12	2.0	30.7251	32.7251	28.7251	2.1614	27.5591	29.5591	-25.5591	0.0993
13	4.0	32.7251	36.7251	28.7251	2.2385	26.5337	30.5337	-22.5337	0.0880
14	6.0	34.7251	40.7251	28.7251	2.3177	25.5605	31.5605	-19.5605	0.0778
15	8.0	36.7251	44.7251	28.7251	2.3991	24.6340	32.6340	-16.6340	0.0685
16	0.0	32.7251	32.7251	32.7251	2.2385	26.5337	26.5337	-26.5337	0.0880
17	2.0	34.7251	36.7251	32.7251	2.3177	25.5605	27.5605	-23.5605	0.0778
18	4.0	36.7251	40.7251	32.7251	2.3991	24.6340	28.6340	-20.6340	0.0685
19	6.0	38.7251	44.7251	32.7251	2.4830	23.7497	29.7497	-17.7497	0.0601
20	8.0	40.7251	48.7251	32.7251	2.5695	22.9035	30.9035	-14.9035	0.0525
21	0.0	36.7251	36.7251	36.7251	2.3991	24.6340	24.6340	-24.6340	0.0685
22	2.0	38.7251	40.7251	36.7251	2.4830	23.7497	25.7497	-21.7497	0.0601
23	4.0	40.7251	44.7251	36.7251	2.5695	22.9035	26.9035	-18.9035	0.0525
24	6.0	42.7251	48.7251	36.7251	2.6589	22.0918	28.0918	-16.0918	0.0458
25	0.0	40.7251	40.7251	40.7251	2.5695	22.9035	22.9035	-22.9035	0.0525
26	2.0	42.7251	44.7251	40.7251	2.6589	22.0918	24.0918	-20.0918	0.0458
27	4.0	44.7251	48.7251	40.7251	2.7515	21.3117	25.3117	-17.3117	0.0397
28	0.0	44.7251	44.7251	44.7251	2.7515	21.3117	21.3117	-21.3117	0.0397
29	2.0	46.7251	48.7251	44.7251	2.8474	20.5605	22.5605	-18.5605	0.0343
30	0.0	48.7251	48.7251	48.7251	2.9470	19.8357	19.8357	-19.8357	0.0295

Problem 17. –The values of the flow angle, α, the Mach angle, μ, and the angles of the characteristics, $\alpha \pm \mu$, for all points of the previous problem are shown in Table P14.17. Compute the slopes m_I and m_{II} and the x,y coordinates for each of the points.

The table below contains the computed data. The Mach angles were computed in the previous problem from the determined Mach number.

Point	μ	$\alpha + \mu$	$\alpha - \mu$	m_I	m_{II}	x	y
1	33.7490	33.7490	-33.7490	-0.6682	0.6682	0.1497	0.0000
2	32.3339	34.3339	-30.3339	-0.5851	0.6756	0.1595	0.0067
3	31.0204	35.0204	-27.0204	-0.5100	0.6918	0.1695	0.0136
4	29.7940	35.7940	-23.7940	-0.4409	0.7109	0.1797	0.0208
5	28.6433	36.6433	-20.6433	0.1405	0.7324	0.3562	0.1501
6	31.0204	31.0204	-31.0204	-0.5932	0.6013	0.1707	0.0000
7	29.7940	31.7940	-27.7940	-0.5185	0.6106	0.1822	0.0070
8	28.6433	32.6433	-24.6433	-0.4498	0.6302	0.1939	0.0144
9	27.5591	33.5591	-21.5591	-0.3859	0.6519	0.3850	0.1389
10	26.5337	34.5337	-18.5337	0.1405	0.6757	0.4133	0.1581
11	28.6433	28.6433	-28.6433	-0.5366	0.5462	0.1952	0.0000
12	27.5591	29.5591	-25.5591	-0.4685	0.5566	0.2086	0.0075
13	26.5337	30.5337	-22.5337	-0.4050	0.5784	0.4149	0.1268
14	25.5605	31.5605	-19.5605	-0.3452	0.6020	0.4474	0.1463
15	24.6340	32.6340	-16.6340	0.1405	0.6272	0.4814	0.1676
16	26.5337	26.5337	-26.5337	-0.4887	0.4993	0.2239	0.0000
17	25.5605	27.5605	-23.5605	-0.4254	0.5106	0.4462	0.1135
18	24.6340	28.6340	-20.6340	-0.3659	0.5339	0.4832	0.1332
19	23.7497	29.7497	-17.7497	-0.3094	0.5587	0.5222	0.1550
20	22.9035	30.9035	-14.9035	0.1405	0.5850	0.5635	0.1792
21	24.6340	24.6340	-24.6340	-0.4473	0.4586	0.7000	0.0000
22	23.7497	25.7497	-21.7497	-0.3877	0.4704	0.7573	0.0270
23	22.9035	26.9035	-18.9035	-0.3312	0.4948	0.8180	0.0570
24	22.0918	28.0918	-16.0918	-0.2773	0.5205	0.8827	0.0907
25	22.9035	22.9035	-22.9035	-0.4107	0.4225	0.8229	0.0000
26	22.0918	24.0918	-20.0918	-0.3541	0.4348	0.8930	0.0305
27	21.3117	25.3117	-17.3117	-0.3000	0.4600	0.9682	0.0650
28	21.3117	21.3117	-21.3117	-0.3779	0.3901	0.9736	0.0000
29	20.5605	22.5605	-18.5605	-0.3237	0.4027	1.0608	0.0351
30	19.8357	19.8357	-19.8357	-0.3482	0.3607	1.1615	0.0000

Table P14.17

Problem 18. –Repeat Problem 14.16 using the region-to-region method for the regions shown in Figure P14.18.

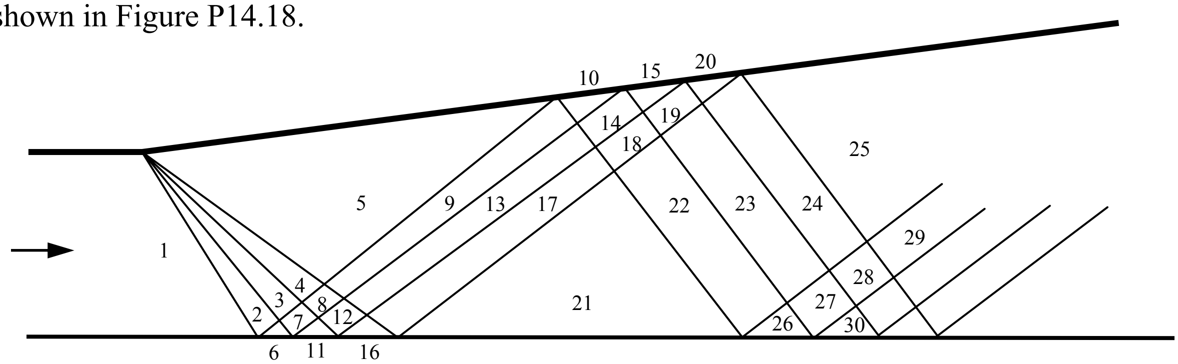
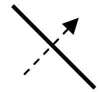
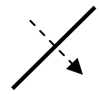
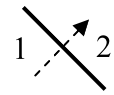


Figure P14.18

Region-to-Region Methodology:

crossing a type I characteristic: $\nu + \alpha = \mathrm{I}$  $\Delta\nu = \Delta\alpha$

crossing a type II characteristic: $\nu - \alpha = \mathrm{II}$  $\Delta\nu = -\Delta\alpha$

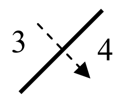
Given: α_1, ν_1, and α_2 Find: ν_2 

$$\Delta\nu = \Delta\alpha$$

or

$$\nu_2 - \nu_1 = \alpha_2 - \alpha_1$$

so $$\boxed{\nu_2 = \alpha_2 + (\nu_1 - \alpha_1) = \alpha_2 + \mathrm{II}_1}$$

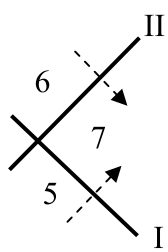
Given: α_3, ν_3, and α_4 Find: ν_4 

$$\Delta\nu = -\Delta\alpha$$

or

$$\nu_4 - \nu_3 = \alpha_3 - \alpha_4$$

so $$\boxed{\nu_4 = -\alpha_4 + (\nu_3 + \alpha_3) = -\alpha_4 + \mathrm{I}_3}$$

Given: α_5, ν_5, and α_5, ν_5  Find: ν_7 and α_7

crossing I between regions 5 and 7: $\nu_7 - \nu_5 = \alpha_7 - \alpha_5$ or $\nu_7 - \alpha_7 = \nu_5 - \alpha_5$

crossing II between regions 6 and 7: $\nu_7 - \nu_6 = -(\alpha_7 - \alpha_6)$ or $\nu_7 + \alpha_7 = \nu_6 + \alpha_6$

solving these two equations simultaneously gives

$$\boxed{\begin{aligned} \nu_7 &= \frac{(\nu_6 + \alpha_6) + (\nu_5 - \alpha_5)}{2} = \frac{\mathrm{I}_6 + \mathrm{II}_5}{2} \\ \alpha_7 &= \frac{(\nu_6 + \alpha_6) - (\nu_5 - \alpha_5)}{2} = \frac{\mathrm{I}_6 - \mathrm{II}_5}{2} \end{aligned}}$$

Method: Region-to-Region								
Region	α	ν	**I = ν+α**	**II = ν−α**	**M**	μ	α + μ	α − μ
1	0.0	20.7251	20.7251	20.7251	1.8000	33.7490	33.7490	-33.7490
2	2.0	22.7251	24.7251	20.7251	1.8697	32.3339	34.3339	-30.3339
3	4.0	24.7251	28.7251	20.7251	1.9405	31.0204	35.0204	-27.0204
4	6.0	26.7251	32.7251	20.7251	2.0125	29.7940	35.7940	-23.7940
5	8.0	28.7251	36.7251	20.7251	2.0861	28.6433	36.6433	-20.6433
6	0.0	24.7251	24.7251	24.7251	1.9405	31.0204	31.0204	-31.0204
7	2.0	26.7251	28.7251	24.7251	2.0125	29.7940	31.7940	-27.7940
8	4.0	28.7251	32.7251	24.7251	2.0861	28.6433	32.6433	-24.6433
9	6.0	30.7251	36.7251	24.7251	2.1614	27.5591	33.5591	-21.5591
10	8.0	32.7251	40.7251	24.7251	2.2385	26.5337	34.5337	-18.5337
11	0.0	28.7251	28.7251	28.7251	2.0861	28.6433	28.6433	-28.6433
12	2.0	30.7251	32.7251	28.7251	2.1614	27.5591	29.5591	-25.5591
13	4.0	32.7251	36.7251	28.7251	2.2385	26.5337	30.5337	-22.5337
14	6.0	34.7251	40.7251	28.7251	2.3177	25.5605	31.5605	-19.5605
15	8.0	36.7251	44.7251	28.7251	2.3991	24.6340	32.6340	-16.6340
16	0.0	32.7251	32.7251	32.7251	2.2385	26.5337	26.5337	-26.5337
17	2.0	34.7251	36.7251	32.7251	2.3177	25.5605	27.5605	-23.5605
18	4.0	36.7251	40.7251	32.7251	2.3991	24.6340	28.6340	-20.6340
19	6.0	38.7251	44.7251	32.7251	2.4830	23.7497	29.7497	-17.7497
20	8.0	40.7251	48.7251	32.7251	2.5695	22.9035	30.9035	-14.9035
21	0.0	36.7251	36.7251	36.7251	2.3991	24.6340	24.6340	-24.6340
22	2.0	38.7251	40.7251	36.7251	2.4830	23.7497	25.7497	-21.7497
23	4.0	40.7251	44.7251	36.7251	2.5695	22.9035	26.9035	-18.9035
24	6.0	42.7251	48.7251	36.7251	2.6589	22.0918	28.0918	-16.0918
25	8.0	44.7251	52.7251	36.7251	2.7515	21.3117	29.3117	-13.3117
26	0.0	40.7251	40.7251	40.7251	2.5695	22.9035	22.9035	-22.9035
27	2.0	42.7251	44.7251	40.7251	2.6589	22.0918	24.0918	-20.0918
28	4.0	44.7251	48.7251	40.7251	2.7515	21.3117	25.3117	-17.3117
29	6.0	46.7251	52.7251	40.7251	2.8474	20.5605	26.5605	-14.5605
30	0.0	44.7251	44.7251	44.7251	2.7515	21.3117	21.3117	-21.3117

Problem 19. – Compute the supersonic flow past the curved contour of a two-dimensional plug nozzle shown in Figure P14.19a. The contour is shaped so as to produce cancellation of the waves incident on the plug. The nozzle is to provide a flow of air ($\gamma = 1.4$) at Mach 1.9502856. The Mach number at the throat of the nozzle is sonic. Use the region-to-region method for a 5 wave expansion as indicated in Figure 14.19b. Determine the Mach number distribution and the inclinations of the characteristics.

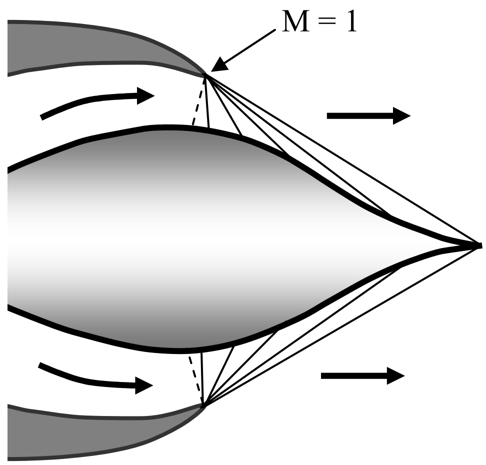


Figure P14.19a

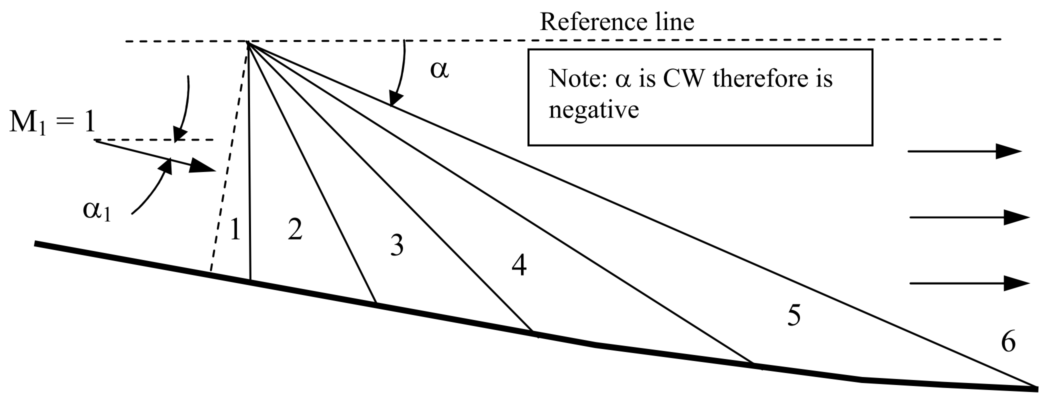


Figure P14.19b

In going from region 1 to region 6 we would have to cross 5 characteristics of Type I for which $\Delta\nu = \Delta\alpha$ or $\nu_6 - \nu_1 = \alpha_6 - \alpha_1$. Since $M_1 = 1$ and $M_6 = 1.9502856$, $\nu_1 = 0$ and $\nu_6 = 25.0000$, respectively. And since $\alpha_6 = 0$, we see that $\alpha_1 = -25.0000^\circ$. Because we are considering the expansion to take place across 5 waves, the flow angle increases by 5˚ in passing from region-to-region. The following table is readily established:

Region	α	ν	**I = ν+α**	**II = ν−α**	**M**	μ	$\alpha + \mu$	$\alpha - \mu$
1	-25.0000	0.0000	-25.0000	25.0000	1.0000	90.0000	65.0000	-115.0000
2	-20.0000	5.0000	-15.0000	25.0000	1.2565	52.7383	32.7383	-72.7383
3	-15.0000	10.0000	-5.0000	25.0000	1.4350	44.1769	29.1769	-59.1769
4	-10.0000	15.0000	5.0000	25.0000	1.6047	38.5474	28.5474	-48.5474
5	-5.0000	20.0000	15.0000	25.0000	1.7750	34.2904	29.2904	-39.2904
6	0.0000	25.0000	25.0000	25.0000	1.9503	30.8469	30.8469	-30.8469

To compute the contour of the surface, we average the slopes of adjoining regions and obtain the following

Region	Region	Inclinat'n
1	2	-93.8692
2	3	-65.9576
3	4	-53.8621
4	5	-43.9189
5	6	-35.0686

Problem 20. – A thin airfoil has the form of a circular arc, as shown in Figure P14.20. Use segregated supersonic flow along a curved surface to determine the lift and drag coefficients for the foil at a Mach number of 1.851177. Take γ = 1.4 and divide the circular arc into 5 linear pieces of equal length. A characteristic will emerge from each of the corners of these lengths on both the upper and lower sides on the foil.

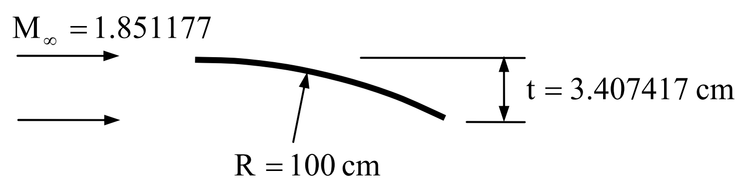
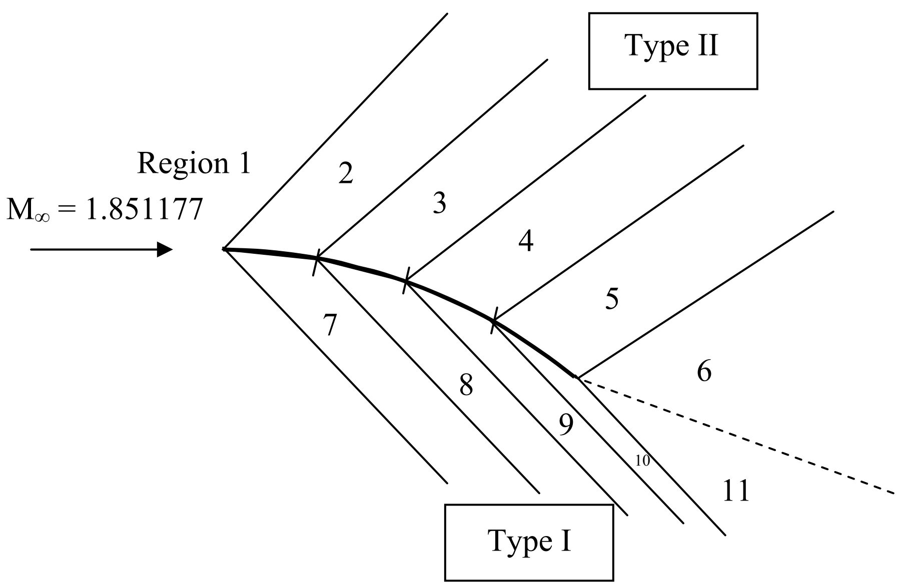


Figure P14.20

The numbering of the regions is contained in the following sketch



Before performing the characteristic calculations, various geometric calculations must be made. Development of the relations is straightforward. The symbols are labeled in the sketch below.

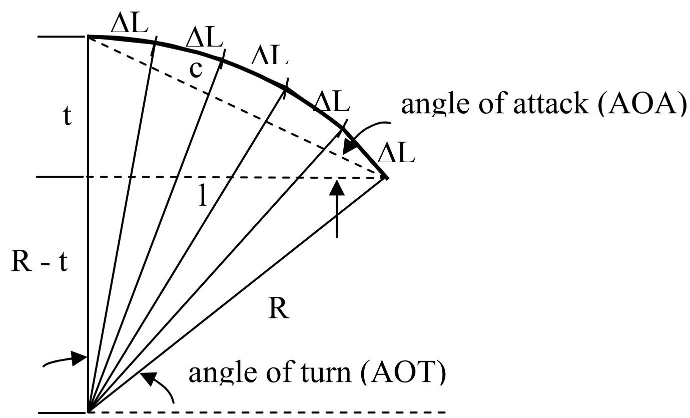
$$AOT = -\cos^{-1}\left(\frac{R-t}{R}\right)$$

$$l = -R\sin(-AOT)$$

$$c = \sqrt{t^2 + l^2}$$

$$\Delta L = -\frac{R(AOT)}{n}$$

$$AOA = \tan^{-1}\left(\frac{t}{R}\right)$$



Input data and the initial calculations for the problem are contained in the following:

γ	α_1	n	$M_\infty = M_1$	R	t
1.4	0	5	1.851177	100	3.407417

ν_1	p_1/p_o	AOT	$\Delta\alpha = \alpha_4 /n$	l	c	ΔL	AOA
22.1970	0.16090	-15	-3.0000	25.8819	26.1052	5.2360	7.5000

FREESTREAM				
Region	**α deg**	**ν deg**	**M**	**p/p_o**
1	0.0	22.1970	1.8512	0.1609

Following the region-to-region procedure (see the solution to Problem 18)

UPPER SURFACE					LOWER SURFACE				
Region	**α deg**	**ν deg**	**M**	**p/p_o**	**Region**	**α deg**	**ν deg**	**M**	**p/p_o**
2	-3.0	25.1970	1.9573	0.1366	7	-3.0	19.1970	1.7474	0.1886
3	-6.0	28.1970	2.0665	0.1152	8	-6.0	16.1970	1.6452	0.2200
4	-9.0	31.1970	2.1794	0.0966	9	-9.0	13.1970	1.5438	0.2556
5	-12.0	34.1970	2.2966	0.0804	10	-12.0	10.1970	1.4417	0.2962
6	-15.0	37.1970	2.4187	0.0664	11	-15.0	7.1970	1.3371	0.3431

Next the pressure difference across the airfoil, i.e., the pressure on the upper surface is subtracted from the pressure on the lower surface, is determined

$(p_{lower} - p_{upper})$	
Segment	$\Delta p/p_o$
1 (R7 - R2)	0.05201
2 (R8 - R3)	0.10474
3 (R9 - R4)	0.15900
4 (R10 - R5)	0.21579
5 (R11 - R6)	0.27662

The lift and drag forces are computed from

$$\frac{\text{Lift}}{p_1} = \frac{p_o}{p_1}\sum_{i=1}^{5}\left(\frac{\Delta p}{p_o}\right)_i \Delta L \cos\alpha_i$$

$$\frac{\text{Drag}}{p_1} = \frac{p_o}{p_1}\sum_{i=1}^{5}\left(\frac{\Delta p}{p_o}\right)_i \Delta L \sin\alpha_i$$

The lift and drag coefficients are computed from

$$C_L = \frac{\text{Lift}}{\frac{\rho_1 V_1^2 c}{2}} = \frac{\text{Lift}/p_1}{\frac{\gamma}{2}\frac{V_1^2}{\gamma p_1/\rho_1}c} = \frac{\text{Lift}/p_1}{\frac{\gamma}{2}M_1^2 c}$$

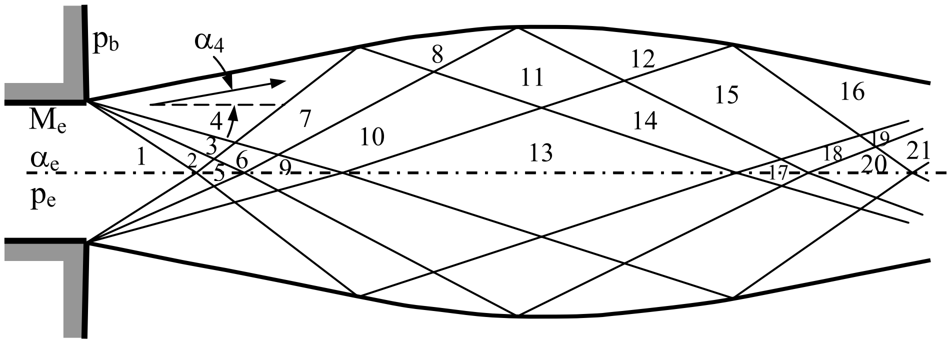
$$C_D = \frac{\text{Drag}}{\frac{\rho_1 V_1^2 c}{2}} = \frac{\text{Drag}/p_1}{\frac{\gamma}{2}M_1^2 c}$$

The results of the calculations are

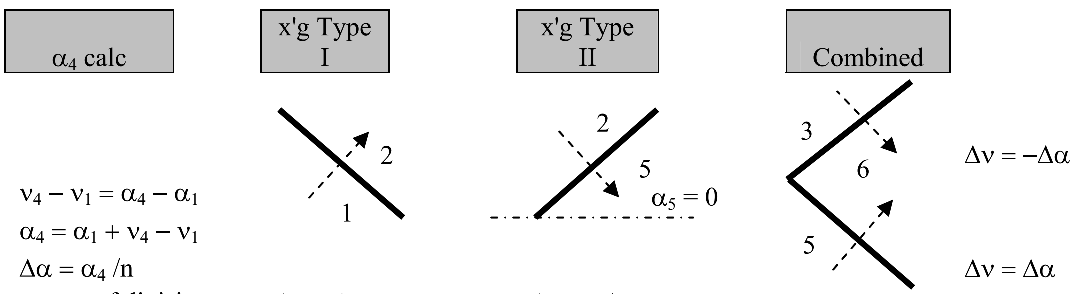
Lift/p_∞	Drag/p_∞	C_L	C_D
0.25754	0.05044	0.41126	0.08055

Problem 21. –A converging-diverging nozzle discharges a uniform supersonic flow at Mach 2.2 and static pressure of 101 kPa two dimensionally into a back pressure region of 69.28701 kPa. Use the Region-to-Region method to determine the flow just downstream of the nozzle exit for the same configuration as employed in Example 14.4. Assume $\gamma = 1.4$.

Because this problem follows that of Example 14.4 for the same configuration, the figure of that example is repeated below



The calculation procedure for the region-to-region method is given as



α_4 calc

$\nu_4 - \nu_1 = \alpha_4 - \alpha_1$

$\alpha_4 = \alpha_1 + \nu_4 - \nu_1$

$\Delta\alpha = \alpha_4 / n$

n = no of divisions

x'g Type I

$\Delta\nu = \Delta\alpha$

$\nu_2 - \nu_1 = \alpha_2 - \alpha_1$

$\nu_2 = \nu_1 + \Delta\alpha$

x'g Type II

$\Delta\nu = -\Delta\alpha$

$\nu_5 - \nu_2 = -(\alpha_5 - \alpha_2)$

$\nu_5 = \nu_2 + \alpha_2$

Combined

$\nu_6 = [(\nu_{5} - \alpha_{5}) + (\nu_3 + \alpha_3)]/2$

$\alpha_6 = \nu_6 - (\nu_5 - \alpha_5)$

The initial and computed data for this problem follows

γ	α_1	$p_e = p_1$	$p_b = p_4$	$M_e = M_1$	p_e/p_o	p_4/p_o	M_4	$\Delta\alpha = \alpha_4 / n$
1.4	0	101	69.28701	2.2	0.09352	0.06416	2.4410	2.0000
n				ν_1			ν_4	
3				31.7325			37.7325	

Because Region 4 is a uniform flow region bordering the free surface: p_4 = 69.28701 kPa. For isentropic flow at M_1 = 2.2 and γ = 1.4,

$$\frac{p_4}{p_o} = \left(\frac{p_4}{p_1}\right)\left(\frac{p_1}{p_o}\right) = \left(\frac{69.28701}{101}\right)(0.09352) = 0.06416$$

So using the isentropic flow solver for this pressure ratio we find $M_4 = 2.4410$ as shown above.

At this Mach number, from the *Prandtl-Meyer Spreadsheet Solver* (PMSS): $\nu_4 = 37.7325°$ and $\mu_4 = \sin^{-1}(1/M_4) = 24.1836°$. Also at $M_1 = 2.2$, the PMSS gives $\nu_1 = 31.7325°$ and $\mu_1 = \sin^{-1}(1/M_1) = 27.0357°$. Hence,

$$\alpha_4 = \alpha_1 + \nu_4 - \nu_1 = 0.0 + 37.7325 - 31.7325 = 6.0000^\circ$$

As seen in Figure 14.16, the expansion fan has been divided into 3 equal pieces so that

$$\alpha_2 = \frac{6.0}{3} = 2.0^\circ, \qquad \alpha_3 = 4.0^\circ$$

The results of the calculations are listed below

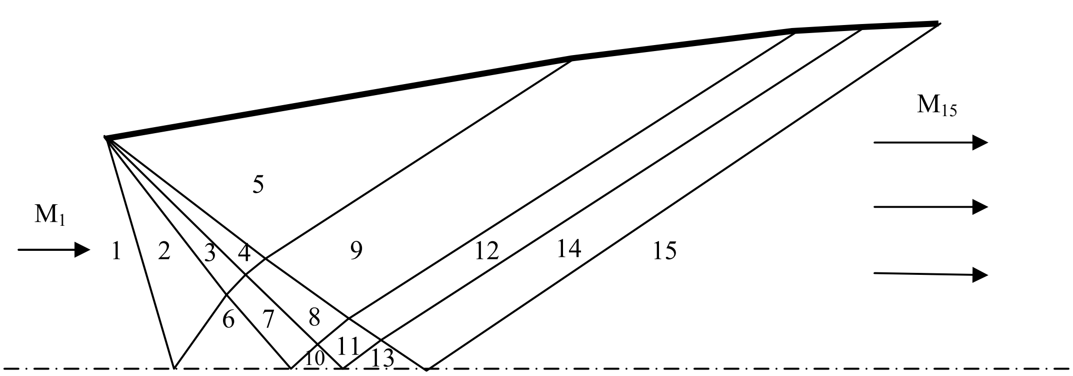
Region	α deg	ν deg	M	μ deg	$\alpha + \mu$ deg	$\alpha - \mu$ deg
1	0.0	31.7325	2.2	27.0357	27.0357	-27.0357
2	2.0000	33.7325	2.2781	26.0373	28.0373	-24.0373
3	4.0000	35.7325	2.3584	25.0883	29.0883	-21.0883
4	6.0000	37.7325	2.4410	24.1836	30.1836	-18.1836
5	0.0	35.7325	2.3584	25.0883	25.0883	-25.0883
6	2.0000	37.7325	2.4410	24.1836	26.1836	-22.1836
7	4.0000	39.7325	2.5262	23.3189	27.3189	-19.3189
8	2.0000	37.7325	2.4410	24.1836	26.1836	-22.1836
9	0.0	39.7325	2.5262	23.3189	23.3189	-23.3189
10	2.0000	41.7325	2.6142	22.4905	24.4905	-20.4905
11	0.0000	39.7325	2.5262	23.3189	23.3189	-23.3189
12	-2.0000	37.7325	2.4410	24.1836	22.1836	-26.1836
13	0.0	43.7325	2.7051	21.6951	21.6951	-21.6951
14	-2.0000	41.7325	2.6142	22.4905	20.4905	-24.4905
15	-4.0000	39.7325	2.5262	23.3189	19.3189	-27.3189
16	-6.0000	37.7325	2.4410	24.1836	18.1836	-30.1836
17	0.0	39.7325	2.5262	23.3189	23.3189	-23.3189
18	-2.0000	37.7325	2.4410	24.1836	22.1836	-26.1836
19	-4.0000	35.7325	2.3584	25.0883	21.0883	-29.0883
20	0.0	39.7325	2.5262	23.3189	23.3189	-23.3189
21	-2.0000	37.7325	2.4410	24.1836	22.1836	-26.1836

The averaged angles of inclination and the slopes of the characteristics are

Type I: $\alpha - \mu$			Type II: $\alpha + \mu$		
Regions	**Angle deg**	**slope$_I$**	**Regions**	**Angle deg**	**slope$_{II}$**
1 - 2	-25.5365	-0.4778	2 - 5	26.5628	0.5000
2 - 3	-22.5628	-0.4155	3 - 6	27.6359	0.5236
3 - 4	-19.6359	-0.3568	4 - 7	28.7513	0.5486
5 - 6	-23.6359	-0.4376	6 - 9	24.7513	0.4610
6 - 7	-20.7513	-0.3789	7 - 10	25.9047	0.4857
7 - 8	-20.7513	-0.3789	8 - 11	24.7513	0.4610
9 - 10	-21.9047	-0.4021	10 - 13	23.0928	0.4264
10 - 11	-21.9047	-0.4021	11 - 14	21.9047	0.4021
11 - 12	-24.7513	-0.4610	12 - 15	20.7513	0.3789
13 - 14	-23.9394	-0.4440	14 - 17	21.9047	0.4021
14 - 15	-23.0928	-0.4264	15 - 18	20.7513	0.3789
15 - 16	-25.9047	-0.4857	16 - 19	19.6359	0.3568
17 - 18	-28.7513	-0.5486	18 - 20	22.7513	0.4194
18 - 19	-26.7513	-0.5041	19 - 21	21.6359	0.3967
20 - 21	-24.7513	-0.4610			

Problem 22. –Repeat Example 14.5 using the region-to-region method. Compare the results.

The numbering and layout of the regions is contained in the following sketch



Input and the maximum turning angle are contained in the following table

γ	α_1	M_1	M_{15}	divisions	$\alpha_{w,max,MLN}$
1.2	0	1	1.8	4	12.151243

Using this information and the region-to-region methodology explained in Problem 18 we can determine the values in the table which follows

Region	α	ν	$I = \nu+\alpha$	$II = \nu-\alpha$	M	μ	$\alpha + \mu$	$\alpha - \mu$
1	0.0000	0.0000	0.0000	0.0000	1.0000	90.0000	90.0000	-90.0000
2	3.0378	3.0378	6.0756	0.0000	1.1659	59.0617	62.0995	-56.0239
3	6.0756	6.0756	12.1512	0.0000	1.2727	51.7871	57.8627	-45.7115
4	9.1134	9.1134	18.2269	0.0000	1.3682	46.9605	56.0740	-37.8471
5	12.1512	12.1512	24.3025	0.0000	1.4583	43.2931	55.4444	-31.1419
6	0.0000	6.0756	6.0756	6.0756	1.2727	51.7871	51.7871	-51.7871
7	3.0378	9.1134	12.1512	6.0756	1.3682	46.9605	49.9984	-43.9227
8	6.0756	12.1512	18.2269	6.0756	1.4583	43.2931	49.3687	-37.2175
9	9.1134	15.1891	24.3025	6.0756	1.5454	40.3208	49.4342	-31.2074
10	0.0000	12.1512	12.1512	12.1512	1.4583	43.2931	43.2931	-43.2931
11	3.0378	15.1891	18.2269	12.1512	1.5454	40.3208	43.3586	-37.2830
12	6.0756	18.2269	24.3025	12.1512	1.6309	37.8175	43.8931	-31.7418
13	0.0000	18.2269	18.2269	18.2269	1.6309	37.8175	37.8175	-37.8175
14	3.0378	21.2647	24.3025	18.2269	1.7156	35.6540	38.6918	-32.6162
15	0.0000	24.3025	24.3025	24.3025	1.8000	33.7490	33.7490	-33.7490

The angles of inclinations of the characteristics can be used to determine the x,y locations of characteristics. These are determined by averaging the characteristic angles, $\alpha \pm \mu$, of adjoining regions.

Type	I	
Region	**Region**	**Inclinat'n**
1	2	-73.0120
2	3	-50.8677
3	4	-41.7793
4	5	-34.4945
6	7	-47.8549
7	8	-40.5701
8	9	-34.2124
10	11	-40.2881
11	12	-34.5124
13	14	-35.2168

Type	II	
Region	**Region**	**Inclinat'n**
2	6	56.9433
3	7	53.9305
4	8	52.7214
5	9	52.4393
7	10	46.6457
8	11	46.3637
9	12	46.6637
11	13	40.5880
12	14	41.2924
14	15	36.2204

Chapter Fifteen

MEASUREMENTS IN COMPRESSIBLE FLOW

Problem 1. – A Pitot tube is placed in a uniform air flow of Mach 2.5. If the Pitot tube indicates a pressure of 500 kPa, find the static pressure of the flow. Take $\gamma = 1.40$.

From the *Rayleigh-Pitot formula*, Eq. (15.7), we have

$$\frac{p_{o2}}{p_1} = \frac{\gamma+1}{2} M_1^2 \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)} \right]^{\frac{1}{\gamma-1}} \qquad \textbf{(15.7)}$$

For an $M_1 = 2.5$ and $\gamma = 1.4$, the pressure ratio is computed to be

$$\frac{p_{o2}}{p_1} = 7.5 \left(\frac{36.0}{34.2} \right)^{2.5} = 8.526136$$

Because $p_{o2} = 500$ kPa, $p_1 = 500/8.52616 = 58.643213$ kPa

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Problem 2. – A Pitot tube is placed in a uniform helium flow. If the Pitot tube indicates a pressure of 280 kPa and the static pressure of the flow is measured to be 20 kPa, find the Mach number. Take $\gamma = 1.40$.

This is the same type of problem as in Example 15.1. Thus, many of the same steps are repeated herein. The first step is to compute the critical pressure ratio, i.e., Eq.(15.1) at M = 1 and $\gamma = 1.4$,

$$\left(\frac{p_o}{p} \right)_{critical} = \left(\frac{\gamma+1}{2} \right)^{\frac{\gamma}{\gamma-1}} = 1.89293$$

If the actual pressure ratio p_{o2}/p_1 is below the critical value, a subsonic Mach number is computed from Eq.(15.2); whereas, if the pressure ratio is above the critical value, we must extract the supersonic Mach number from the Rayleigh-Pitot formula, Eq.(15.7). To accomplish this we will use the Newton-Raphson procedure that is easily incorporated into a spreadsheet program. Equation (15.7) may be written in the following form

$$f(M) = AM^{2\gamma} - BM^2 + C = 0$$

where the coefficients in this expression are

$$A = \frac{(\gamma+1)^{\gamma+1}}{2^{\gamma-1}}, \qquad B = 4\gamma\left(\frac{p_{o2}}{p_1}\right)^{\gamma-1}, \qquad C = 2(\gamma-1)\left(\frac{p_{o2}}{p_1}\right)^{\gamma-1}$$

The derivative of this function is

$$\frac{df}{dM}(M) = 2\gamma AM^{2\gamma-1} - 2BM$$

The Netwon-Raphson algorithm is

$$M_{new} = M_{old} - \frac{f(M_{old})}{\frac{df}{dM}(M_{old})} = \frac{(2\gamma-1)AM_{old}^{2\gamma} - BM_{old}^2 - C}{2\left(\gamma AM_{old}^{2\gamma-1} - BM_{old}\right)}$$

For this case p_{o2} = 280 kPa and p_1 = 20 kPa, so the pressure ratio is 14.000, which is well above the critical pressure ratio for the given ratio of specific heats. It should be noted that the computed coefficients for this case are

$$A = 6.195766$$
$$B = 16.093083$$
$$C = 2.3722561$$

The results of the iterative computations are presented in following table

n	M(old)	f(M)	df/dM	M(new)
1	4.00000	45.32248	81.61429	3.44467
2	3.44467	9.08901	49.86759	3.26241
3	3.26241	0.84077	40.75026	3.24178
4	3.24178	0.01024	39.75931	3.24152
5	3.24152	0.00000	39.74700	3.24152

Rayleigh-Pitot formula computations

Hence, the computed Mach number for this case is $M_\infty = M_1 = 3.241522$.

Problem 3. – A uniform flow of air at Mach 2.0 passes over an insulated wall. The static temperature and pressure in the free stream outside the boundary layer are, respectively, 250 K and 20 kPa. Determine the free-stream stagnation temperature, adiabatic wall temperature, and static pressure at the wall surface. Take $\gamma = 1.40$.

From Eq.(15.13) we have

$$r \equiv \frac{T_{aw} - T_\infty}{T_{o\infty} - T_\infty} \qquad \textbf{(15.13)}$$

Since $T_\infty = 250K$ and $M_\infty = 2.0$

$$T_{o\infty} = T_\infty\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right) = 250\left(1 + \frac{4}{5}\right) = 450K$$

Assuming a turbulent boundary layer of air (Pr = 0.72)

$$r = \sqrt[3]{Pr} = \sqrt[3]{0.72} = 0.896281$$

$$T_{aw} = T_\infty + r\left(T_{o\infty} - T_\infty\right) = 250 + 0.896281(450 - 250) = 429.25619K$$

The static pressure at the wall is the same as the free stream static pressure: 20kPa

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Problem 4. – A total temperature probe is inserted into the flow of Problem 3. If the probe has K [see Eq.(15.16)] equal to 0.97, what temperature will be indicated by the probe?

From Eq.(15.16)

$$K = \frac{T_{o,indicated} - T_\infty}{T_{o\infty} - T_\infty} \qquad \textbf{(15.16)}$$

So

$$T_{o,indicated} = T_\infty + K\left(T_{o\infty} - T_\infty\right) = 250 + 0.97\left(450 - 250\right) = 444.0K$$

Problem 5. – Sketch a plot of p/p_o versus M for isentropic flow. On the same coordinates, plot p_∞/p_{Pitot} versus M_∞. Take $\gamma = 1.40$.

Values are computed for a range of Mach numbers from Eq.(15.7) for the Raleigh Pitot formula

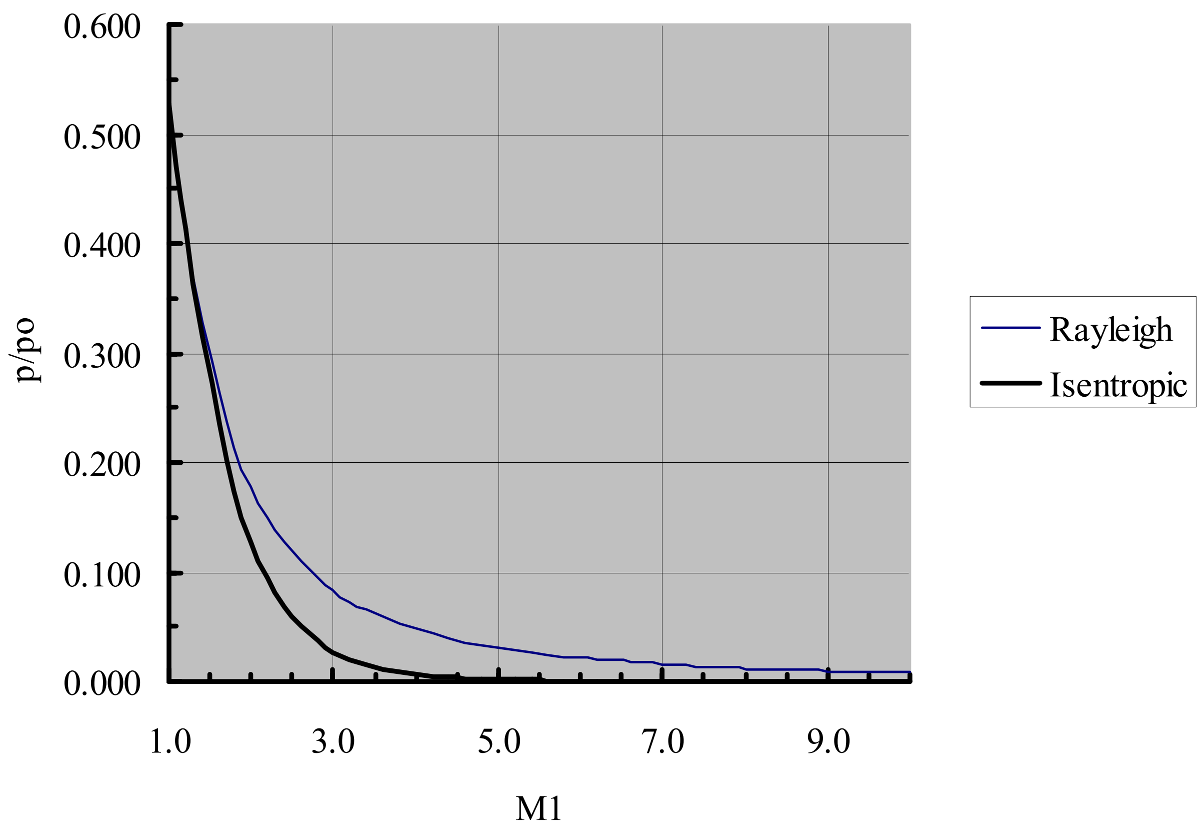
$$\frac{p_{o2}}{p_1} = \frac{\gamma + 1}{2} M_1^2 \left[\frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)}\right]^{\frac{1}{\gamma - 1}} \qquad \textbf{(15.7)}$$

and from Eq.(15.1) for the isentropic relation

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}} \qquad \textbf{(15.1)}$$

The computed values appear in the following table and the accompanying chart

	Rayleigh	isentropic
M_1	**p_1/p_{o2}**	**p/p_o**
1.0	0.528282	0.528282
1.2	0.415368	0.412377
1.4	0.327951	0.314241
1.6	0.262814	0.235271
1.8	0.214155	0.174040
2.0	0.177291	0.127805
2.2	0.148888	0.093522
2.4	0.126632	0.068399
2.6	0.108917	0.050115
2.8	0.094613	0.036848
3.0	0.082912	0.027224
3.2	0.073228	0.020228
3.4	0.065129	0.015125
3.6	0.058290	0.011385
3.8	0.052465	0.008629
4.0	0.047465	0.006586
4.2	0.043143	0.005062
4.4	0.039381	0.003918
4.6	0.036088	0.003053
4.8	0.033189	0.002394
5.0	0.030625	0.001890
5.2	0.028345	0.001501
5.4	0.026310	0.001200
5.6	0.024485	0.000964
5.8	0.022843	0.000779
6.0	0.021361	0.000633
6.2	0.020017	0.000517
6.4	0.018797	0.000425
6.6	0.017684	0.000350
6.8	0.016667	0.000290
7.0	0.015735	0.000242
7.2	0.014879	0.000202
7.4	0.014091	0.000169
7.6	0.013363	0.000143
7.8	0.012691	0.000121
8.0	0.012068	0.000102
8.2	0.011489	0.000087
8.4	0.010952	0.000075
8.6	0.010450	0.000064
8.8	0.009983	0.000055
9.0	0.009546	0.000047
9.2	0.009137	0.000041
9.4	0.008754	0.000036
9.6	0.008395	0.000031
9.8	0.008057	0.000027
10.0	0.007739	0.000024



Problem 6. – Derive the Gladstone-Dale relation, Eq.(15.26), from the Lorenz-Lorentz relation, Eq.(15.25).

From Eq.(15.25) we have

$$\frac{n^2-1}{n^2+2}=C\rho \qquad \textbf{(15.25)}$$

For values of n near unity we may write that n = 1 + ε, where ε is very small. Therefore,

$$n^2-1=(1+\varepsilon)^2-1=2\varepsilon+\varepsilon^2 \cong 2\varepsilon$$

$$n^2+2=(1+\varepsilon)^2+2=3+2\varepsilon+\varepsilon^2 \cong 3$$

So

$$\frac{n^2-1}{n^2+2} \cong \frac{2\varepsilon}{3} = \frac{2}{3}(n-1) = C\rho$$

Hence,

$$n = 1 + \frac{3}{2}C\rho = 1 + K\rho$$

the *Gladstone-Dale equation* is obtained.

Problem 7. – Compute the index of refraction at atmospheric pressure for the gases contained in Table 15.1 for the given Gladstone-Dale constants and temperatures.

To use the Gladstone-Dale equation, we must first compute the density of each gas assuming each behaves as a perfect gas

Gas	**T (K)**	**R (kJ/kg·K)**	**p (kPa)**	**ρ (m^3/kg)**	**K (cm^3/g)**	**n**
He	295	2.077	101.3	0.16533	0.196	1.0000324
H_2	273	4.124	101.3	0.08998	1.55	1.0001395
O_2	273	0.2598	101.3	1.42826	0.19	1.0002714
N_2	273	0.2968	101.3	1.25021	0.238	1.0002975
CO_2	295	0.1889	101.3	1.81784	0.229	1.0004163

Problem 8. – The wire of a hot wire anemometer is placed to an air flow at atmospheric pressure with a temperature of 30°C and a velocity of 80 m/s. The wire is heated to a constant temperature of 210°C. The diameter of the wire is 4 μm and its length is 2 mm. Determine the electric current in the wire. The air properties at the mean film temperature are: ρ = 0.898 kg/m^3, μ = $2.27 \cdot 10^{-5}$ kg/m·s, k=0.0328 W/m·K, and c_p=1.013 kJ/kg·K. The resistivity of the wire is 0.22 μΩ·m.

The Reynolds number is:

$$Re = \frac{\rho V_\infty d}{\mu} = \frac{(0.8980 kg/m^3)(80 m/s)(4 \cdot 10^{-6} m)}{2.27 \cdot 10^{-5} kg/m \cdot s} = 12.66$$

The Prandtl number is:

$$Pr = \frac{\mu c_p}{k} = \frac{(2.27 \cdot 10^{-5} kg/m \cdot s)(1.013 \cdot 10^3 J/kg \cdot K)}{0.0328 W/m \cdot K} = 0.70$$

With Prandtl and Reynolds numbers we can determine the Nusselt number from Kramers correlation, Eq.(15.23):

$$\mathrm{Nu} = 0.42\,\mathrm{Pr}^{0.2} + 0.57\,\mathrm{Pr}^{0.33}\,\mathrm{Re}^{0.5} = 0.42 \cdot 0.70^{0.2} + 0.57 \cdot 0.70^{0.33} \cdot 12.66^{0.5} = 2.19$$

The heat transfer coefficient can then be calculated as:

$$h = \frac{\mathrm{Nu} \cdot k}{d} = \frac{2.19 \cdot (0.0328\,\mathrm{W/m \cdot K})}{4 \cdot 10^{-6}\,\mathrm{m}} = 17958\,\mathrm{W/m^2 \cdot K}$$

The heat loss from the wire is given by:

$$q = h \cdot A \cdot (T_w - T_\infty) = h \cdot (\pi \cdot d \cdot L) \cdot (T_w - T_\infty)$$

$$= (17958\,\mathrm{W/m^2 \cdot K}) \cdot \left[\pi \cdot (4 \cdot 10^{-6}\,\mathrm{m}) \cdot (2 \cdot 10^{-3}\,\mathrm{m})\right] \cdot (210 - 30)\mathrm{K} = 0.08124\,\mathrm{W}$$

The resistance of the wire is:

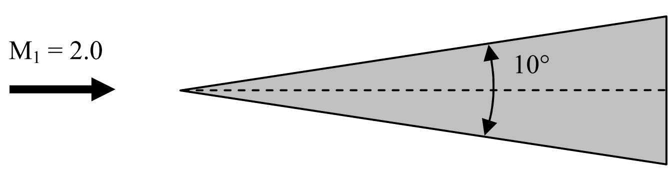
$$R = \frac{\rho_w L}{(\pi d^2/4)} = \frac{(0.22 \cdot 10^{-6}\,\Omega \cdot \mathrm{m})(2 \cdot 10^{-3}\,\mathrm{m})}{\left[\pi (4 \cdot 10^{-6}\,\mathrm{m})^2/4\right]} = 35\Omega$$

Consequently, the electric current in the wire is:

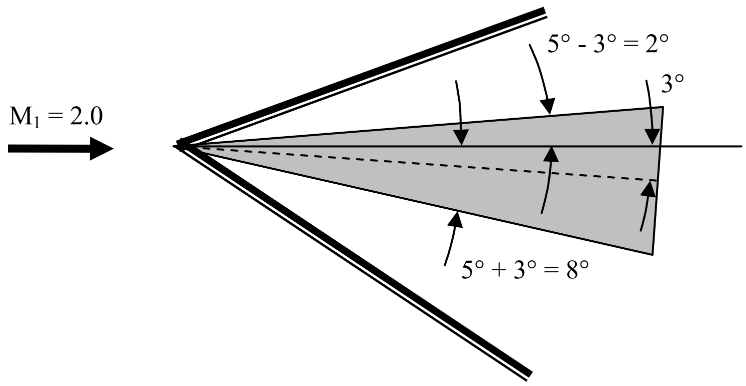
$$I = \sqrt{\frac{q}{R}} = \sqrt{\frac{0.08124\,\mathrm{W}}{35\Omega}} = 0.048\,\mathrm{A} = 48\,\mathrm{mA}$$

Problem 9. – A symmetrical wedge of 10° total included angle is placed in a uniform Mach 2.0 flow of static pressure of 60 kPa. If the axis of the wedge is misaligned with the flow direction by 3°, determine the static pressure difference between the top and bottom surfaces of the wedge. Take $\gamma = 1.40$

The symmetrical wedge is shown as follows



The misaligned wedge is shown below along with the various angles of deflection. With these angles, the upstream Mach number and the ratio of specific heats, it is a simple matter using the oblique shock solver developed in Chapter 6 to determine the flow characteristics shown in the following tables:



Upper Surface

Given:	M_1 and δ			
Weak Shock Solution				
γ	M_1	δ(deg)	θ(deg)	p_2/p_1
1.4000	2.0000	2.0000	31.6463	1.1180
		0.0349	1.6225	67.07914
		radians	cotθ	

Lower Surface

Given:	M_1 and δ			
Weak Shock Solution				
γ	M_1	δ(deg)	θ(deg)	p_2/p_1
1.4000	2.0000	8.0000	37.2101	1.5400
		0.1396	1.3170	92.39894
		radians	cotθ	

The pressure difference between the lower surface and the upper surface is then

$$\Delta p = p_{lower} - p_{upper} = 92.39894 - 67.07914 = 25.3198 kPa$$

Problem 10. – The temperature of the wire of an anemometer placed perpendicular to an air flow of 20°C is 100°C. The wire dissipates 20 mW of heat to the flow. The diameter of the wire is 3 μm and the length 1 mm. What is the velocity of the flow? The properties of the air at the mean temperature between fluid and the wire are: ρ = 1.0595 kg/m^3, c_p = 1.009 kJ/kg, k = 0.0285 W/m·K, and $\mu = 2\cdot10^{-5}$ kg/m·s.

The heat transfer coefficient is:

$$h=\frac{q}{A\left(T_w-T_\infty\right)}=\frac{q}{\pi dL\left(T_w-T_\infty\right)}=\frac{20\cdot 10^{-3}\,W}{\pi\cdot\left(3\cdot 10^{-6}\,m\right)\left(1\cdot 10^{-3}\right)\left(100-20\right)K}=26525.8W/m^2\cdot K$$

The Nusselt number is computed from:

$$Nu=\frac{hd}{k}=\frac{\left(26525.8W/m^2\cdot K\right)\left(3\cdot 10^{-6}\,m\right)}{0.0285W/m\cdot K}=2.79$$

The Prandtl number is:

$$Pr=\frac{\mu c_p}{k}=\frac{\left(2\cdot 10^{-5}\,kg/m\cdot s\right)\left(1.009\cdot 10^3\,J/kg\cdot K\right)}{0.0285W/m\cdot K}=0.708$$

Therefore, the Reynolds number is determined from Kramers correlation, Eq.(15.26):

$$Re=\left(\frac{Nu-0.42\cdot Pr^{0.2}}{0.57\cdot Pr^{0.33}}\right)^2=\left(\frac{2.79-0.42\cdot 0.708^{0.2}}{0.57\cdot 0.708^{0.33}}\right)^2=22.2$$

The velocity of the flow is therefore:

$$v=\frac{Re\,\mu}{\rho d}=\frac{22.2\cdot\left(2.\cdot 10^{-5}\,kg/m\cdot s\right)}{\left(1.0595kg/m^3\right)\cdot\left(3\cdot 10^{-6}\,m\right)}=139.6m/s$$

Problem 11. – The sensing element of a hot wire anemometer is a platinum wire 4 μm diameter and 2 mm length. The wire is placed perpendicular to an air flow at atmospheric pressure with a temperature of 20°C and a velocity of 60 m/s. If the temperature of the wire is 100°C, determine the power dissipated by the wire. The properties of the air at the mean film temperature are: ρ = 1.0595 kg/m^3, c_p = 1.009 kJ/kg·K, k = 0.0285 W/m·K, $\mu = 2\cdot 10^{-5}$ kg/m·s.

The Reynolds number is:

$$Re=\frac{\rho V_\infty d}{\mu}=\frac{\left(1.0595kg/m^3\right)\left(60m/s\right)\left(4\cdot 10^{-6}\,m\right)}{2\cdot 10^{-5}\,kg/m\cdot s}=12.714$$

The Prandtl number is:

$$Pr=\frac{\mu c_p}{k}=\frac{\left(2\cdot 10^{-5}\,kg/m\cdot s\right)\left(1.009\cdot 10^3\,J/kg\cdot K\right)}{0.0285W/m\cdot K}=0.708$$

The Nusselt number can be computed using the Kramers correlation, Eq.(15.23):

$$Nu = 0.42\,Pr^{0.2} + 0.57\,Pr^{0.33}\,Re^{0.5} = 0.42 \cdot 0.708^{0.2} + 0.57 \cdot 0.708^{0.33} \cdot 12.714^{0.5} = 2.2$$

The heat transfer coefficient is:

$$h = \frac{Nu \cdot k}{d} = \frac{2.2 \cdot (0.0285 W / m \cdot K)}{4 \cdot 10^{-6} m} = 15675 W / m^2 \cdot K$$

The heat loss from the wire is:

$$q = h \cdot A \cdot (T_w - T_\infty) = h \cdot (\pi \cdot d \cdot L) \cdot (T_w - T_\infty) =$$

$$= (15675 W / m^2 \cdot K) \cdot \left[\pi \cdot (4 \cdot 10^{-6} m) \cdot (2 \cdot 10^{-3} m)\right] \cdot (100 - 20) K = 0.0315 W = 31.5 mW$$

Problem 12. – A dual beam LDV-system with a wavelength of 3000Å and a 20° angle between the intersecting beams records a difference of 30MHz between the two Doppler shifts. What is the velocity of the flow-field?

The amplitude of the wave vector is

$$K = \frac{4\pi \sin\kappa}{\lambda} = \frac{(4\pi)(\sin 10^\circ)}{3000 \cdot 10^{-10} m} = 7273757.86\ m^{-1}$$

The velocity of the seeding particle (assumed equal to the velocity of the flow) is

$$v = \frac{\nu_D \cdot 2\pi}{K} = \frac{(30 \cdot 10^6 Hz) \cdot (2\pi)}{7273757.86 m^{-1}} = 25.914\ \ m/s$$

Problem 13. – What is the minimum frequency that a dual beam LDV system has to have in order to measure a 1000 m/s velocity with a 5000 Å laser with a 5° angle between the beams? Explain using both theoretical explanations of the LVD instrument.

Using the Doppler shift explanation: the frequency difference between the two Doppler shifts is

$$\nu_D = \frac{vK}{2\pi}$$

where the amplitude of the wave vector is

$$K = \frac{4\pi \sin\kappa}{\lambda}$$

so the frequency to detect is

$$\nu_D = \frac{2v \sin\kappa}{\lambda}$$

Alternately, using the fringe model: the fringe spacing distance is

$$d_f = \frac{\lambda}{2\sin\kappa}$$

so the frequency to detect can be determined, as above, from,

$$\nu_D = \frac{v}{d_f} = \frac{2v\sin\kappa}{\lambda} = \frac{(2\cdot 1000\,\mathrm{m/s})\cdot(\sin 2.5^0)}{5000\cdot 10^{-10}\,\mathrm{m}} = 174.478\ \text{MHz}$$

Problem 14. – Determine the vorticity of the flowfield based on the double-exposed PIV photograph shown Figure P15.14. The interval between the two exposures is Δt = 0.001s. The grid-size equals 1mm in both directions.

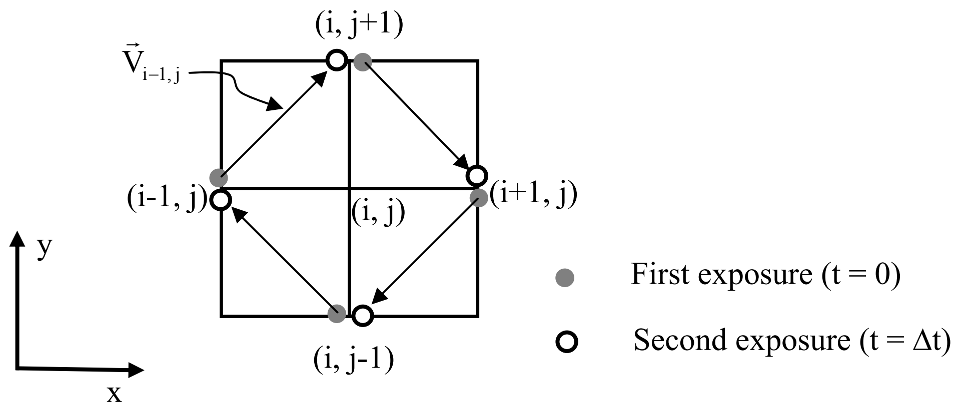


Figure P15.14

The vorticity of the flow is defined as

$$\vec{\zeta} = \nabla \times \vec{V}$$

For two-dimensional flow,

$$\zeta_{i,j} = \left(\frac{\partial v}{\partial x}\right)_{i,j} - \left(\frac{\partial u}{\partial y}\right)_{i,j}$$

The vorticity can be numerically approximated using central differences as

$$\zeta_{i,j} \cong \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$

From Figure P15.14

$$v_{i+1,j} \cong \frac{-\Delta y}{\Delta t} = \frac{-0.001\text{m}}{0.001\text{s}} = -1 \quad \text{m/s}$$

Similarly,

$$v_{i-1,j} \cong \frac{\Delta y}{\Delta t} = 1 \text{ m/s}, \quad u_{i,j+1} \cong \frac{\Delta x}{\Delta t} = 1 \text{ m/s}, \quad u_{i,j-1} \cong \frac{-\Delta x}{\Delta t} = -1 \text{ m/s}$$

Hence, the vorticity is

$$\zeta \cong \frac{-1-1}{2\cdot 10^{-3}} - \frac{1-(-1)}{2\cdot 10^{-3}} = -2\cdot 10^{3} \text{ s}^{-1}$$

Problem 15. – A double–exposed PIV photograph contains the flowfield illustrated in Figure P15.15. Show that if the grid-size is equal in the x and y directions, i.e., $\Delta x = \Delta y$, the vorticity at (i,j) is only dependent upon the time interval Δt between the two exposures.

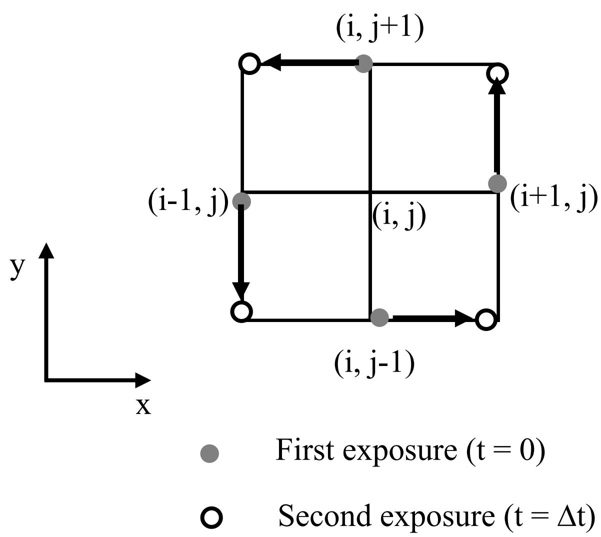


Figure P15.15

The vorticity can be numerically calculated from

$$\zeta_{i,j} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{i,j} \cong \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$

Now,

$$v_{i+1,j} = \frac{\Delta y}{\Delta t}\ ;\ v_{i-1,j} = -\frac{\Delta y}{\Delta t}\ ;\ u_{i,j+1} = -\frac{\Delta x}{\Delta t}\ ;\ u_{i,j-1} = \frac{\Delta x}{\Delta t}$$

Hence, for $\Delta x = \Delta y$ we have

$$\zeta_{i,j} = \frac{2}{\Delta t}$$