

NONLINEAR OPTICS OF SOLENOID MAGNETS*

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Abstract

Solenoid magnets are often employed for focusing in low energy beam transport lattices in the front-end of a machine. We derive a relatively simple analytic formula for the nonlinear angular focusing kick imparted to particles traversing the solenoid. Few approximations are made. The formula provides a clear expression of the well-known *Scherzer's Theorem* from electron optics that all nonlinear terms of an axisymmetric magnetic field provide more focusing and that nonlinear terms are non-removable. The formula suggests that for beam transport, little can be done to reduce nonlinearities in solenoid-type magnets other than take a simple design without abrupt changes as a function of axial coordinate and appropriately choose the aspect ratio (characteristic bore radius over axial length) of the magnet system and the beam filling factor within the aperture to limit nonlinear effects. Illustrative applications of the formula characterize nonlinear focusing effects in iron-free and iron type solenoid magnets.

INTRODUCTION

Solenoid magnets are appealing due to their simple field structure in idealized form. However, as the level of detail of the model description increases, the cross-coupled structure of the equations of motion describing particle focusing in a solenoid become increasingly complicated and difficult to analyze. In this study, we derive a closed analytic formula for the radial angular impulse imparted to a particle traversing an axisymmetric solenoid magnet including both linear and *all* nonlinear field components. The only approximation made is the paraxial approximation. The formula applies to particles with arbitrary (conserved) canonical angular momentum. This is important because in many cases particles are born from a source immersed in a magnetic field, and consequently, the particles can have finite canonical angular momentum. Near the source, particles are typically transported downstream using an axisymmetric solenoid focusing lattice where the particle canonical angular momentum is conserved. The surprisingly simple formula obtained for the focusing impulse provides insight on the nonlinear focusing properties of solenoid magnets. The formula is applied to better understand nonlinear focusing effects in solenoids, both as the radius of the particle within the magnet aperture increases and as the aspect ratio and structure of the solenoid is varied. The formula should be possible to exploit in future studies to estimate how statisti-

cal rms beam emittance (quality) evolves during transport in a nonlinear solenoid lattice.

DERIVATION

We consider the a single particle of charge q and mass m evolving in an axisymmetric ($\partial/\partial\theta = 0$) applied solenoid field \mathbf{B}^a and employ r, θ, z cylindrical-polar coordinates. Because the magnetic field only bends particle trajectories, the kinetic energy of the particle is constant, or equivalently, the exact relativistic gamma factor ($\gamma = 1/\sqrt{1-\beta^2} = \text{const}$) and beta factors ($\beta = \text{const}$) are conserved. The applied magnetic field of the solenoid is assumed axisymmetric, so the particle evolves with a conserved angular momentum $P_\theta \equiv [\mathbf{x} \times (\mathbf{p} + q\mathbf{A})] \cdot \hat{\mathbf{z}} = r(p_\theta + qA_\theta) = \text{const}$. Here, \mathbf{x} and $\mathbf{p} = m\gamma\dot{\mathbf{x}}$ are the coordinate and mechanical momentum of the particle, $\dot{} \equiv \frac{d}{dt}$ denotes a derivative with respect to the time t , and \mathbf{A} is the vector potential that generates the applied field (i.e., $\mathbf{B}^a = \nabla \times \mathbf{A}$). Applying Stoke's theorem to the expression for P_θ obtains the so-called Bush's theorem expression [1, 2]

$$P_\theta = \gamma m r^2 \dot{\theta} + \frac{q\psi}{2\pi} = \text{const}. \quad (1)$$

Here, $\psi = \int_r d^2x B_z^a(r, \theta) = \oint_r \mathbf{A} \cdot d\ell = 2\pi r A_\theta$ is the magnetic flux bounded by a circle of radius r . In general, one cannot take $P_\theta = 0$ for orbits that do not go through the origin $r = 0$ (i.e., P_θ cannot in general be zeroed by coordinate choice for all particles) which corresponds to the axis of symmetry of the solenoid.

The static field Maxwell equations $\nabla \cdot \mathbf{B}^a = 0$ and $\nabla \times \mathbf{B}^a = 0$ allow $\mathbf{B}^a = \hat{\mathbf{r}}B_r(r, z) + \hat{\mathbf{z}}B_z(r, z)$ to be expanded as [1, 2]

$$B_r(r, z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{r}{2}\right)^{2\nu-1}, \quad (2)$$

$$B_z(r, z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu},$$

where $B_{z0}(z) \equiv B_z(r=0, z)$ is the on-axis field. Thus, the 3D field of the axisymmetric magnet with all nonlinear terms can be regarded as specified by the on-axis ($z=0$) field. The lowest-order terms in the sums in Eq. (2): $B_r \simeq -\frac{1}{2} \frac{\partial B_{z0}}{\partial z} r$ and $B_z \simeq B_{z0}$ provide linear focusing. All higher terms provide nonlinear focusing [1, 2].

The radial component of the Lorentz force equation for the particle evolving in the magnetic field $\ddot{\mathbf{x}} = [q/(\gamma m)]\mathbf{x} \times \mathbf{B}^a$ can be expressed as

$$\ddot{r} = r\dot{\theta}^2 - \frac{qr\dot{\theta}B_z^a}{\gamma m}. \quad (3)$$

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Taking the paraxial approximation with $\dot{r} \approx \beta^2 c^2 r''$ in Eq. (3) (exactly one has: $\ddot{z} = -\frac{qr\dot{\theta}B_z^a}{\gamma m}$ and $\ddot{r} = \dot{z}^2 r'' - \frac{qr\dot{\theta}B_z^a}{\gamma m} r'$) and then applying Eqs. (1) and (2) and considerable manipulation obtains the radial equation of motion:

$$r'' - \frac{P_\theta^2}{\gamma^2 \beta^2 m^2 c^2 r^3} \approx - \sum_{n=0}^{\infty} r^{2n+1} \frac{(-1)^n}{2^{2n+2}} \sum_{\nu=0}^n \frac{2n-2\nu+1}{\nu!(\nu+1)!(n-\nu)!(n-\nu+1)!} \left[\frac{B_{z0}^{(2\nu)}(z)}{[B\rho]} \right] \left[\frac{B_{z0}^{(2n-2\nu)}(z)}{[B\rho]} \right] + \frac{P_\theta}{\gamma \beta m c} \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{2^{2\nu} (\nu-1)!(\nu+1)!} \left[\frac{B_{z0}^{(2\nu)}(z)}{[B\rho]} \right] r^{2\nu-1}. \quad (4)$$

Here, $[B\rho] \equiv \gamma \beta m c / q = \text{const}$ is the particle rigidity and $B_{z0}^{(n)}(z)$ denotes the n th derivative of B_{z0} with respect to z [e.g., $B_{z0}^{(2)} = B_{z0}''$].

The angular impulse imparted to an energetic particle that traverses the solenoid field at nearly constant radius measures the focusing strength (interpreted as a nonlinear thin-lens impulse). We define

$$\Delta r'|_{\text{focus}} \equiv \int_{-\infty}^{\infty} dz \left[r'' - \frac{[P_\theta/(\gamma \beta m c)]^2}{r^3} \right] \approx \int_{-\infty}^{\infty} dz \text{ [RHS Eq. (4)] } \Big|_{r \text{ fixed}} \quad (5)$$

and obtain

$$\Delta r'|_{\text{focus}} = - \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^{n+1} (n!)^2 (n+2)!} \int_{-\infty}^{\infty} dz \left(\frac{B_{z0}^{(n)}(z)}{[B\rho]} \right)^2 r^{2n+1}. \quad (6)$$

In the steps to obtain Eq. (6), all canonical angular momentum terms in Eq. (6) vanish because $\int_{-\infty}^{\infty} dz \frac{\partial^\nu B_{z0}}{\partial z^\nu} = 0$ for $\nu > 0$, partial integrations are applied iteratively to show that $\int_{-\infty}^{\infty} dz \left[\frac{\partial^{2\nu} B_{z0}}{\partial z^{2\nu}} \right] \left[\frac{\partial^{2n-2\nu} B_{z0}}{\partial z^{2n-2\nu}} \right] = (-1)^n \int_{-\infty}^{\infty} dz \left(\frac{\partial^n B_{z0}}{\partial z^n} \right)^2$, and significant simplifications have been made in numerical coefficients. The formula, as derived applies to a single solenoid magnet in isolation with an orbit extending from the beginning to the end of the fringe field. Effectively the same result is obtained in a periodic solenoid lattice (even with overlapping fringe fields) if we replace $\int_{-\infty}^{\infty} dz \dots \implies \int_{\text{Period}} dz \dots$ and interpret the focusing impulse $\Delta r'|_{\text{focus}}$ as applying per lattice period.

The radial form of the angular kick integral in Eq. (5) with the P_θ term subtraction may appear unusual, but it is the appropriate form to take. In analysis of the familiar x -plane Hill's equation $x'' + \kappa(z)x = 0$, it is well-established that $\Delta x'|_{\text{focus}} = - \int_{-\infty}^{\infty} dz \kappa(z)x \approx -x \int_{-\infty}^{\infty} dz \kappa(z)$ provides an impulse measure of strength of the lattice focusing function $\kappa(z)$ [1]. Equation (5) provides the correct impulse measure for orbits in cylindrical-polar coordinates with $P_\theta = \text{const}$. It can be shown that the formula produces the correct results for free-space, continuous focusing [i.e., $x'' + k_{\beta 0}^2 x = 0$ and $y'' + k_{\beta 0}^2 y = 0$ with $k_{\beta 0}^2(z) \geq 0$ specified], and for linear optics approximation magnetic focusing (with fringe field).

DISCUSSION

Note from the impulse equation (6) that the first term gives the familiar Larmor-frame focusing strength with $\Delta r'|_{\text{focus}} = -r \int_{-\infty}^{\infty} dz [k_L(z)]^2$ where $k_L = B_{z0}/(2[B\rho])$ is the Larmor wavenumber [1, 3]. All higher-order terms are negative definite since they are z -integral accumulations of $(B_{z0}^{(n)}/[B\rho])^2$ with a negative numerical coefficient. This shows that *all* nonlinearities contribute *stronger* focusing as the radius of the particle increases and are *not removable*. Also, the sign of B_{z0} is irrelevant for all nonlinear terms showing that the polarity of magnet coils (or permanent magnet material) is irrelevant to all orders (However, in a periodic lattice, if neighboring solenoid magnets overlap the relative polarity of neighboring pairs of magnets can matter). Nonlinear terms may potentially be made larger or smaller by varying the magnet structure in z , but it is not possible to completely eliminate terms since $B_{z0}^{(n)}$ cannot be identically zero for all z . Intuitively, one expects a weaker nonlinear impulse if any axial variations of the solenoid magnet structure are made gradually. Rapid changes will likely to increase on-axis field derivative amplitudes driving more nonlinearity. Thus, one expects methods such as the necking down an iron yoke in the ends of the solenoid magnet or use of "bucking coils" to contain the axial fringe extent to potentially do more harm than good in terms of nonlinear effects by increasing the amplitude in higher derivatives of the on-axis field $B_{z0}(z)$. The best one might do is to provide magnet designers with as small an aspect ratio (characteristic coil radius divided by axial length of magnet structure) as possible, keep the magnet design simple with gradual, if any, tapering, and to limit nonlinear effects through the choice of the fill factor of the beam in the magnet aperture. Not surprisingly, this is consistent with usual front-end applications of solenoids in accelerator systems. Finally, results suggest making a ring with solenoid focused is not a good idea since errors are not removable and can be expected to accumulate lap-by-lap leading to potential rms emittance growth unless the beam fill factor in the solenoids is very small. These points are further illustrated by the results in the following Application section.

Results found as a consequence of Eq. (6) are known in the electron optics community as *Scherzer's Theorem* which is applicable to the focusing of charged particles by axisym-

metric fields (both magnetic, electric, and combined) [4]. The analogous results appear to be derived in eikonal perturbative formulations and have significantly complicated expression. The present result in Eq. (6) has considerably simplified form and follows to all orders within the context of the paraxial approximation. The paraxial approximation is usually very good in charged particle accelerators – though it is weakest near the source (injection energy). Equation (6) can be generalized to include non-paraxial effects, but the expression of results become considerably more complex.

APPLICATIONS

We employ the impulse measure of solenoid focusing strength in Eq. (6) to calculate the ratio of nonlinear ($n \geq 1$ terms) to linear (first $n = 0$ term) focusing terms as

$$\begin{aligned}
 F &\equiv \frac{\Delta r'|_{\text{Nonlin focus terms}}}{\Delta r'|_{\text{Linear focus term}}} \\
 &= \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{n+1}(n!)^2(n+2)!} \frac{\int_{-\infty}^{\infty} dz [B_{z0}^{(n)}(z)]^2}{\int_{-\infty}^{\infty} dz [B_{z0}(z)]^2} r^{2n} \quad (7) \\
 &= \left[\frac{(r/r_c)^2}{3^2 \cdot 2^6} \int_{-\infty}^{\infty} dz [r_c B'_{z0}]^2 + \frac{5(r/r_c)^4}{2^6} \int_{-\infty}^{\infty} dz [r_c^2 B''_{z0}]^2 \right. \\
 &\quad \left. + \frac{7(r/r_c)^6}{3^2 \cdot 2^6} \int_{-\infty}^{\infty} dz [r_c^3 B'''_{z0}]^2 + \dots \right] / \left[\int_{-\infty}^{\infty} dz [B_{z0}]^2 \right].
 \end{aligned}$$

Here, r_c is the closest approach radius of the magnet coils (or material structures). This factor is arbitrary (divides out of terms) but is inserted to illustrate characteristic scaling. Note that $[B\rho]$ divides out of the expression for F , so the fractional error depends *only* on the field structure and *not* the dynamics – thereby simplifying error characterization.

First, we apply Eq. (7) to a simple thin-coil iron-free solenoid design with a coil of NI amp-turns, radius R and axial length ℓ is centered at $z = 0$. The on-axis field is straightforward to calculate analytically from the Biot-Savart law [5] to show that

$$B_{z0}(z) = \frac{\mu_0(NI)}{2\ell} \left[\frac{(z+\ell/2)}{\sqrt{(z+\ell/2)^2 + R^2}} - \frac{(z-\ell/2)}{\sqrt{(z-\ell/2)^2 + R^2}} \right]$$

with $\mu_0 = 4\pi \times 10^{-7} \text{N/A}^2$. Associated integrals in Eq. (7) are surprisingly difficult to express in simple closed analytic form, but are convergent and can be evaluated numerically to plot the nonlinear impulse fraction F as a function of $r/R \in [0, 1]$ for values of magnet aspect ratio R/ℓ . Representative plots of B_{z0} , B'_{z0} , B''_{z0} , and F are given in Fig. 1. Note that F can become relatively large at high fill factors.

Finally, we apply the fractional nonlinear impulse formula (7) to an S4 solenoid magnet in the front-end of the FRIB accelerator [6]. The S4 solenoid is a typical iron yoke magnet with an operating field $|B_{z0}| \sim 1$ Tesla, a clear bore radius of 7.94 cm, and a material (iron) axial length of 39.51 cm. The axial field B_{z0} is generated from a finite element magnet design code. The fractional error impulse evaluated with the first two terms of Eq. (7) is

$$F \simeq 0.125 \left(\frac{r}{r_c} \right)^2 + 0.0278 \left(\frac{r}{r_c} \right)^4 + \Theta \left(\frac{r}{r_c} \right)^6.$$

These two terms are adequate to provide almost full convergence (so noisy numerical high-order derivatives not needed). Typical fill factors of the beam within the magnet aperture are $\sim 50\%$, corresponding to an $\sim 3\%$ nonlinearity (increasing to 8% and 15% at 3/4 and full aperture), which drives negligible emittance growth in simulations [7].

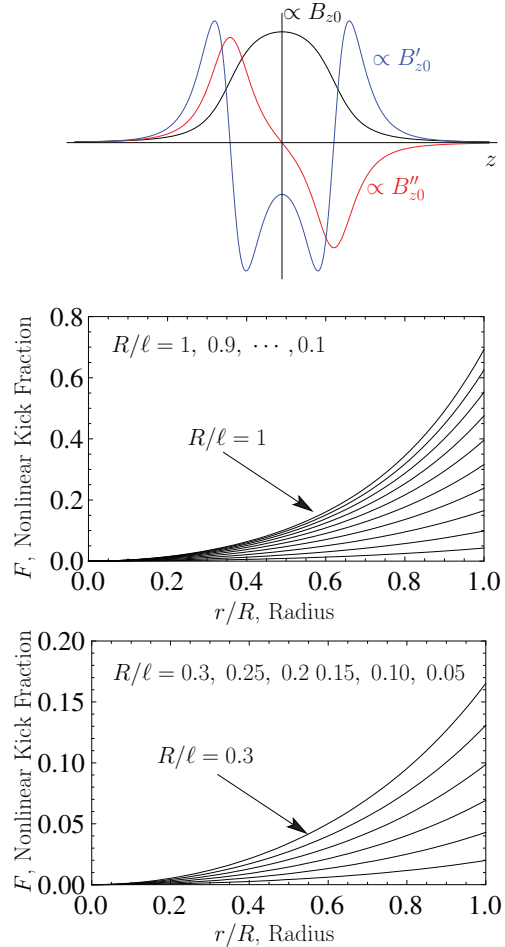


Figure 1: (Color) For a thin-coil iron-free solenoid: (upper) B_{z0} , B'_{z0} , and B''_{z0} (rescaled), and (middle, lower) fractional nonlinear impulse F .

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