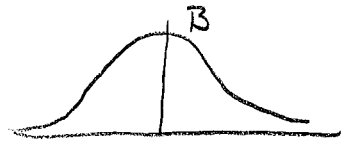
Solenoids Notes

(solenoid in general)

Often a solenoid lens is defined by giving the axial field

$B(z) \equiv B(r=0, z) \Rightarrow$



The reason for this is that you can, in principle, do an expansion for the off-axis field for an axisymmetric lens

$$B_z(r,z) = B(z) - \frac{r^2}{4} B''(z) + \cdots \qquad B_r(r,z) = -\frac{r}{2} B'(r) + \frac{r^3}{16} B'''(z) + \cdots$$

$$B'(z) = \frac{\partial B(z)}{\partial z}$$

Today I am only interested in the first terms. To motivate this:

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \frac{1}{r}\frac{\partial (r B_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0$$

approx

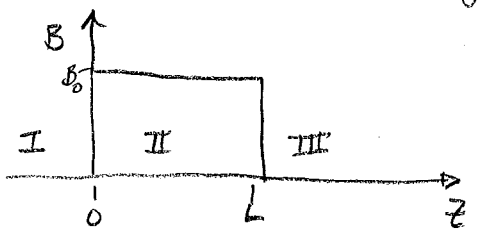
$$\int_0^r \frac{\partial r B_r}{\partial r} dr = -\int_0^r r \left.\frac{\partial B_z}{\partial z}\right|_{z=0} dr$$

$$r B_r(r,z) = -\frac{r^2}{2} B'(z)$$

$$\boxed{B_r(r,z) = -\frac{r}{2} B'(z)}$$

Ok, let's use this to look at particle behavior when passing through a solenoid lens.

→ Simple, then generalize



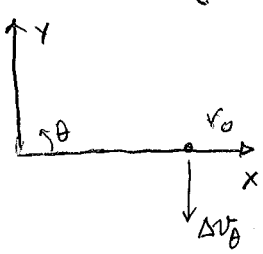
$$B(z) = B_0\left[u(z) - u(z-L)\right] \qquad u(z) = \begin{cases} 1 & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$B_r(z) = -\frac{r}{2} B_0 \left[\delta(z) - \delta(z-L)\right] \qquad \int_a^b \delta(z)\,dz = \begin{cases} 1 & a \leq 0 \leq \\ 0 & \text{otherwise} \end{cases}$$

OK Let's have an ion approach at $\vec{v} = v_z \hat{k}$ $r = r_0$ this solenoid

I. $r = r_0$ $\vec{v} = v_z \hat{k}$

Boundary: $\vec{F} = q\vec{v} \times \vec{B} \Rightarrow \vec{a} = \frac{q}{m} \vec{v} \times \vec{B}$ $\qquad B_z = 0$, $B_r < 0$



$$\vec{a} = \frac{q}{m} v_z \frac{r_0}{2} B_0 \,\delta(z)\,(-\hat{\theta})$$

$$\vec{v} = \vec{v}_0 + \int \vec{a}\,dt = \vec{v}_0 + \int \vec{a} \frac{dt}{dz} dz = \vec{v}_0 + \underbrace{\frac{q}{m}\frac{r_0}{2} B_0 (-\hat{\theta})}_{\Delta v_\theta}$$

$$v_\perp \equiv \Delta v_\theta = -\frac{r_0}{2} \frac{q B_0}{m}$$

Question: Now we are in region 2 with $\vec{B} = B_0 \hat{k}$ with v_z and a kick of $v_\perp$. What does subsequent motion look like

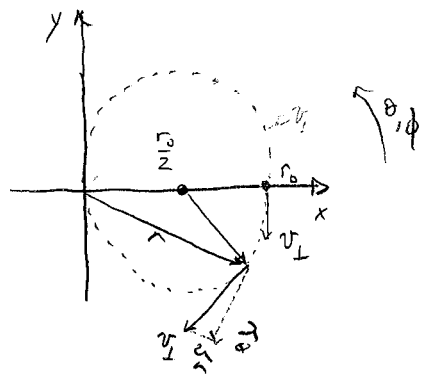
on board

$$B(z) = B_0\left[u(z) - u(z-L)\right]$$

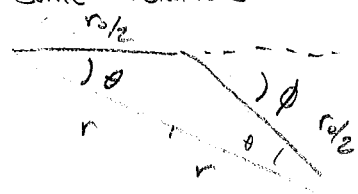
$$B_r(z) = -\frac{r}{2}B_0\left[\delta(z) - \delta(z-L)\right]$$

$I \to II \qquad \Delta v_\theta = -\frac{r_0}{2}\omega_c \qquad v_r = \frac{r_0}{2}\omega_c$

Answer



Some relations



$\frac{1}{2}\phi = \theta \qquad \phi = -\omega_c t =$

$\theta = -\frac{1}{2}\omega_c t =$

$v_r = v_\perp \sin\theta$

$v_\theta = -v_\perp \cos\theta \qquad \cos\theta = \frac{r}{r_0}$

want to write v_r, v_θ in terms of r, not r_0

relations $\theta = -\frac{1}{2}\omega_c t = -\frac{1}{2}\omega_c \frac{z}{v_z}$

$$r = |r_0 \cos\theta| = r_0\left|\cos\frac{\omega_c z}{2 v_z}\right|$$

$$v_\theta = -\frac{\omega_c r_0}{2}\cos\theta = -\frac{\omega_c r}{2} \quad \text{(always negative)}$$

$$v_r = \frac{\omega_c r_0}{2}\sin\theta = \frac{\omega_c r}{2}\tan\theta = \frac{\omega_c r}{2}\tan\left(\frac{\omega_c z}{2 v_z}\right)$$

Now at $z = L$ $(II \to III)$

$$r_1 = r_0\left|\cos\frac{\omega_c L}{2 v_z}\right| \qquad v_\theta = -\frac{\omega_c r_1}{2} \qquad v_r = -\frac{\omega_c r}{2}\tan\left(\frac{\omega_c L}{2 v_z}\right)$$

Another boundary so get another Δv_θ

$I \to II \quad \Delta v_\theta = -\frac{r_0}{2}\omega_c \qquad II \to III \quad \Delta v_\theta = +\frac{r_1}{2}\omega_c$

Exactly cancels with v_θ

so III $\rightarrow$ $v_\theta = 0$ $\quad v_r = -\frac{\omega_c r_1}{2}\tan\left(\frac{\omega_c L}{2 v_z}\right)$

$r_1 = r_0 \cos$

$\downarrow$

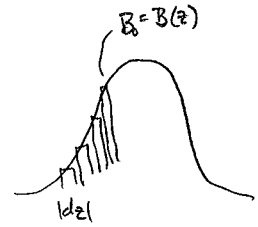
linear in r_0!

for right B's L's focusing linear

2 Results $\quad r_1 = r_0$

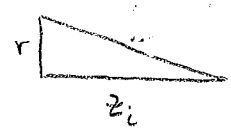
1) $L \ll \frac{v_z}{\omega_c} \rightarrow v_r \simeq -\frac{\omega_c r_0}{2}\left(\frac{\omega_c L}{2 v_z}\right) = -\frac{\omega_c^2 r_0 L}{4 v_z} = -\frac{q^2 B_0^2 r L}{4 m^2 v_z}$

Make a real lens this way



$$v_{r\,tot} = \int dv_r = -\frac{q^2 r}{4 m^2 v_z}\int B^2(z)\,dz$$

$$r' = \frac{v_r}{v_z}$$



$\frac{\Delta r}{\Delta z} = \frac{r}{z_1} = r'$

What is focal length?

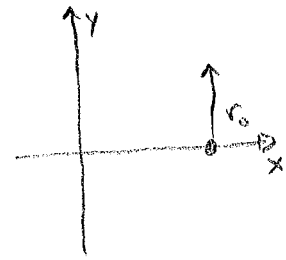
$z_i = \frac{r}{r'} \quad \frac{1}{z_i} = \frac{r'}{r}$

$$\frac{1}{f} = \cancel{\frac{1}{z_0}}^{\,0} + \frac{1}{z_i} = \frac{v_r}{v_z r} = \frac{q^2}{4 m^2 v_z^2}\int B^2(z)\,dz$$

in

What if we start in a region II and leave?

$$\Delta v_\theta = \frac{r_0 \omega_c}{2}$$



After that just drift

Introduces angular momentum $\vec{L} = \vec{r} \times \vec{p}$

At waist $\varepsilon = x_{max} x'_{max} = r_0 \frac{\Delta v_\theta}{v_z} = r_0^2 \frac{q B_0}{2 v_z m}$

Busch Theorem

$$v_z = \sqrt{\frac{2 Q e V_{ext}}{M m_0}}$$

$$\varepsilon \propto r^2 \sqrt{\frac{Q}{M}}$$

increases with $\sqrt{Q}$

not what is seen in an ECR, say

$$\varepsilon \downarrow Q$$

Assume high Q created on axis